Quantum Phase Transitions in Random Spin Systems

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Abstract

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A number of condensed matter systems undergo a phase transition at zero temperature as some external parameter (such as pressure, magnetic field, or amount of dirt) is varied. Quantum effects are often crucial to the physics of this phenomenon - hence the name "quantum phase transitions". This thesis is concerned with a study of such zero temperature phase transitions in the presence of static randomness (due to impurities or other frozen defects in the system). Experimentally accessible quantum phase transitions often occur in the presence of strong randomness, and are very poorly understood. Theoretically, the description of such phenomena involving competition between various kinds of potential energy of interactions, quantum effects, and randomness presents a challenging problem, where there are as yet few reliable techniques. This thesis studies simple quantum statistical models with randomness as a useful starting point to obtain insight into more complex, realistic systems. Progress is reported in understanding various simple but non-trivial models of random quantum magnetic systems in the vicinity of a quantum phase transition. The results show that the effects of randommess may be quite dramatic, and lead to a phenomenology that is strikingly different from that of pure systems.

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Contents

Chapter 1

Introduction

Phase transitions in condensed matter have been of major interest to physicists for many decades. Most research has focused on the properties of transitions that occur at some finite temperature as an external parameter is varied and there now exists a detailed, quantitative understanding of these. An important feature of these finite temperature phase transitions is that a complete description of them can be achieved based entirely on the principles of classical statistical mechanics. While quantum mechanics may be important in determining the existence and properties of the various phases, it is absolutely correct to ignore it to understand the most interesting properties of the transition. In recent years however, a number of systems have been studied where the phase transition occurs at zero temperature. Quantum mechanics plays an important role in the physics of these transitions, often being responsible for their very occurence. For this reason, such zero temperature phase transitions are often called quantum phase transitions.

There exist many experimental systems where such transitions are seen. For instance, there are transitions from an insulator to a metal[1] or a superconductor[2], and various magnetic-nonmagnetic transitions in heavy fermion compounds[3], high-T_c cuprates[4], and so on most of which are poorly understood. Such many body systems on the brink of a ground state instability show an interesting phenomenology at finite temperature quite different from more conventional ones. On the theoretical side, there seem to exist many significant differences from the treatment of ordinary critical phenomena, especially in metallic and/or disordered systems, which make

them interesting.

Most experimental systems are disordered. At sufficiently low temperatures, effects of both randomness and quantum mechanics must be taken into account to provide a proper description. These effects are often particularly important in the vicinity of a $T = 0$ critical point separating two different phases. Such phenomena where randomness, quantum mechanics, and interactions all play an important role represent a class of problems where there are as yet few reliable techniques and are only beginning to be understood.

This thesis is concerned with the study of simple quantum statistical mechanical models with disorder. It is hoped that this will be a useful starting point to obtain insight into the properties of more complex, realistic quantum systems undergoing a $T = 0$ phase transition in the presence of quenched randomness. Even the models considered here turn out to be considerably complicated, and as yet are far from being thoroughly understood. Nevertheless, there are some situations in which it has been possible to make progress and obtain reliable information on universal aspects of the phase transition. These limited available results reveal that the critical properties of random quantum phase transitions may be markedly different from those of pure systems.

The outline of the thesis is as follows: In Chapter 2, we give a brief review of the salient aspects of critical phenomena associated with $T = 0$ transitions in pure quantum systems. This will also be used as an opportunity to introduce the models whose random versions will be studied in later chapters. Chapter 3 is devoted to some general considerations on quantum transitions in random systems. We mention various stray results that are known, present some general arguments to illustrate some of the peculiar properties of random quantum systems as opposed to pure ones, and finally demonstrate that conventional techniques for dealing with critical phenomena (such as the ϵ-expansion) are currently insufficient for the random quantum transitions of interest in this thesis.

The one shining exception to the almost complete absence of reliably understood models of random quantum transitions is the random version of the one dimensional Ising model in a transverse field - perhaps the simplest random quantum system.

This model (or mathematically equivalent versions) have been studied extensively by McCoy and Wu[5], Shankar and Murthy[6], and in particular, by D.S. Fisher[7]. The properties were found to be very unusual as compared to phase transitions in pure quantum systems. One of the main points to be made in this thesis is that there exist a variety of other models, both in one dimension and in higher dimensions, that share many of the unusual properties of the random transverse field Ising chain.

In Chapter 4, we consider the critical properties of a number of random quantum transitions in models (such as the Potts models) with discrete symmetry in $d = 1$. We show that the techniques used by Fisher to solve the Ising problem can also be used to obtain the exact critical properties of all these other models. The results demonstrate that the peculiar properties found for the Ising model also hold for the other models, and hence are not just artifacts of Ising symmetry. Quite remarkably, it turns out that all the computable universal properties of all the models are *identical* to those of the Ising model. This is in striking contrast to the non-random situation where the properties of the transition (and, in certain cases, even whether it is first or second order) depend crucially on the particular symmetry of the model. We call this feature "superuniversality" as it goes well beyond the usual universality expected near second order phase transitions.

In Chapter 5, we address the important question of whether the novel features of the $d = 1$ results can survive in higher dimensions. We show that it is, in principle, possible that they do, by explicitly demonstrating a particular model which displays much of the unusual physics found in $d = 1$. The model we consider is that of an Ising model in a transverse field in a diluted lattice. It turns out that there is a non-trivial quantum transition at the percolation threshold of the lattice whose properties are determined largely by the geometrical properties of the percolating clusters, about which much is known. This enables us to make definitive statements about the critical properties of this transition. To our knowledge, these are the first reliable calculations of the detailed critical properties of a random quantum transition in finite dimension $d > 1$.

Chapter 6 is devoted to a discussion of the situation where the randomness couples directly to the order parameter (random field models). A complete understand-

ing of this problem is lacking, but we provide general scaling hypotheses in close analogy with the corresponding classical problem (which also is not very well understood yet). In particular, we point out various similarities between quantum phase transitions in these random field systems, and the ones considered earlier in the thesis.

In Chapter 7, we discuss the critical properties of quantum spin glass models with infinite range interactions with the q-state Potts model as an example. The critical properties can be found exactly, and hence these models are useful. It turns out, however, that the transition is quite conventional and does not have any of the unusual features found in the models studied in previous chapters. It is not clear yet whether these infinite range models are useful starting points to understand spin glass transitions in realistic models with short range interactions in finite dimensions.

Finally, in Chapter 8, we summarize the principal conclusions of our work, speculate on their more general validity, and comment on their relationship with the work of various other authors.

Chapter 2

Quantum Phase Transitions in Pure Systems

Scaling and renormalization group theory provide the general framework for understanding phase transitions. In the absence of randomness, it has often been possible to implement renormalization group ideas and obtain a fairly general theory of a quantum phase transition. An important and powerful approach in this context, which we will elaborate on below, relies on the possibility of mapping a d-dimensional quantum system at zero temperature to a $d+1$-dimensional *classical* statistical mechanical system at its finite temperature. One can then often use all the knowledge and technology collected on classical statistical mechanical problems to understand the quantum problem. However, it is important to realize that there are some fundamental differences between classical and quantum phase transitions. As is well-known, in classical statistical mechanics, static and dynamic properties can be treated independant of each other. This is however not true in quantum statistical mechanics. Thus it is necessary to treat fluctuations in space and time on an equal footing even to understand thermodynamic properties of a quantum system. (See Ref.[8, 9] for a very clear discussion of this point). Therefore, unlike classical phase transitions, the static and dynamic critical behaviour are intertwined at a quantum phase transition. The extra dimension in the classical system to which a quantum system may formally be mapped indeed reflects the need to include temporal fluctuations to describe the properties of the latter. As in general, the character of the

fluctuations in time may be quite different from the spatial ones, the corresponding classical system may well be considerably anisotropic with respect to the extra dimension.

In this chapter, we will outline the general theory of quantum phase transitions in pure systems. We will be brief as there already exist several excellent reviews[8, 9]. For concreteness, we will for the most part base the discussion on a single model - the Ising model in a transverse field. This is the prototypical example of a system undergoing a quantum transition, and many of its properties are quite generic to quantum transitions in pure systems. We will also introduce and discuss the properties of other quantum models whose random versions will be studied in later chapters.

2.1 The transverse field Ising model

Consider the system defined by the Hamiltonian

$$H = \underbrace{-J \sum_{<i,j>} \sigma_i^z \sigma_j^z}_{H_0} \underbrace{- h \sum_i \sigma_i^x}_{H_1} \tag{2.1}$$

where the $\vec{\sigma_i}$ are Pauli spin matrices placed on the sites i of a d-dimensional hypercubic lattice. When $h = 0$, this is just the usual Ising Hamiltonian. In the presence of a non-zero h, the two terms in the Hamiltonian do not commute, and we get an Ising system with non-trivial quantum fluctuations. The first term may be considered as a potential energy of interaction between the Ising spins which tends to point spins at neighbouring sites along the same direction (we have assumed $J > 0$). The second term, on the other hand, corresponds to a kinetic energy which tends to flip a spin pointing in the $+z$ direction to $-z$, and vice versa. Clearly the Hamiltonian is invariant under a Z_2 symmetry: $\sigma_i^z \rightarrow -\sigma_i^z$.

First, let us consider the ground states of the Hamiltonian in some extreme limits. If $J \gg h$, the first term in Eqn.2.1 dominates, and we may think of the system perturbatively from the $h = 0$ limit. Exactly at $h = 0$, the Hamiltonian is trivially diagoanlized. There are two degenerate ground states- with all spins either

up or down. The lowest excitation about either ground state corresponds to a single spin flip and costs energy $\sim J$. For $h \neq 0$, but small, perturbation theory converges, and we get a ground state with broken symmetry and LRO in σ_z. The lowest-lying excitations can be roughly thought of as a band of delocalized single spin flips. The lowest excitation gap decreases as h increases but is nevertheless non-zero so long as the ground state is ordered.

Similarly, in the opposite limit, when $J \ll h$, we may think of the system perturbatively from the $J = 0$ point. At $J = 0$, different sites decouple, and at each site the spin points along the $+x$ direction. So there is no broken symmetry and long range order(LRO). The lowest excitations about this simple ground state correspond to flipping a single spin to the $-x$ direction, and cost an energy $2h$. For $J \neq 0$ but $\ll h$, perturbation theory converges and we have a paramagnetic ground state with no broken symmetry and LRO. The low-lying excitations can now be roughly thought of as corresponding to a band of delocalized $-x$ spins. The lowest excitation gap, while smaller than $2h$, is still non-zero.

Clearly these two kinds of ground states in these two extreme limits must be separated by a phase transition. As we will see, this transition is second order. Close to this transition, the length scales characterizing the decay of spatial correlations in either phase become very large and ultimately diverge at the critical point. Likewise the energy gap in either phase becomes very small, vanishing exactly at the critical point. Many properties of the system will be universal (*i.e* independant of microscopic details) in this region, and show scaling behaviour characteristic of critical points.

It is useful at this point to introduce the mapping to a classical $(d+1)$-dimensional Ising model. Consider the partition function at a temperature $T = \frac{1}{\beta}$:

$$Z = Tr(e^{-\beta H}) \tag{2.2}$$

The operator $e^{-\beta H}$ can be regarded as the evolution operator from (imaginary) time 0 to β. Using standard techniques, we may write Z as a sum over various possible (imaginary time) histories of the system:

$$Z = Tr(\underbrace{e^{-\epsilon H} e^{-\epsilon H} \ldots e^{-\epsilon H}}_{N\,factors}) \qquad (\epsilon = \frac{\beta}{N})$$

$$= \sum_{\sigma_i^z(\tau)} \langle \sigma_i^z(0)|e^{-\epsilon H}|\sigma_i^z(N-1)\rangle\langle \sigma_i^z(N-1)|e^{-\epsilon H}|\sigma_i^z(N-2)\rangle$$
$$\ldots\ldots \langle \sigma_i^z(1)|e^{-\epsilon H}|\sigma_i^z(0)\rangle$$

where we have introduced at each time step τ, a complete set of states $\sigma_i^z(\tau)$. We take N large so that ϵ is very small. Then we may write

$$e^{-\epsilon H} = e^{-\epsilon(H_0+H_1)} \simeq e^{-\epsilon H_0} e^{-\epsilon H_1}$$

Consider any matrix element in the product above:

$$\begin{aligned} \langle \sigma_i^z(\tau+1)|e^{-\epsilon H}|\sigma_i^z(\tau)\rangle &\simeq \langle \sigma_i^z(\tau+1)|e^{-\epsilon H_0} e^{-\epsilon H_1}|\sigma_i^z(\tau)\rangle \\ &\sim e^{\epsilon J \sum_{<ij>} \sigma_i^z(\tau+1)\sigma_j^z(\tau+1) + h^* \sum_i \sigma_i^z(\tau+1)\sigma_i^z(\tau)} \end{aligned}$$

where $\tanh(h^*) = e^{-2\epsilon h}$. At $T = 0$, clearly we may interpret the partition function expressed as a sum over products of such matrix elements as the partition function of a classical Ising model in $(d+1)$-dimensions with bond strengths in the d-spatial directions equal to ϵJ, and the bond strength in the extra "time" direction equal to h^*. By universality arguments, we then expect that the global properties of the phase diagram and the phase transitions of the quantum model at $T = 0$ are the same as those of an isotropic classical $(d+1)$-dimensional Ising model at finite temperatures. In particular, universal critical properties of the quantum model close to its $T = 0$ phase transition can be directly obtained from the properties of the isotropic classical model near its finite temperature transition[1].

2.1.1 Scaling and renormalization group theory

In this section we will discuss the properties of the transverse field Ising model near its zero temperature phase transition from the point of view of scaling and renormalization group theory. Whenever appropriate, we will also point out qualitative

[1] We caution that this feature is a very special property of the Ising model and some other simple models considered later in this chapter. In general, the anisotropy between the spatial and time directions cannot be ignored and leads to differences in the way spatial and temporal correlations scale

features that may be special to this model, and comment on the more general situation. On approaching the critical point (from either side), there is a correlation length ξ which diverges as

$$\xi \sim |r - r_c|^{-\nu} \tag{2.3}$$

where $r = \frac{h}{J}$ is the control parameter that needs to be tuned to place the system at the critical point. The mapping to the classical model described in the previous section implies that the exponent ν is the same as that of the classical Ising model in one higher dimension. There also is an energy scale Δ that vanishes as

$$\Delta \sim |r - r_c|^{z\nu} \tag{2.4}$$

thereby defining the exponent z. In the particular case of the Ising model being considered here, this energy scale is just the energy gap in either phase. More generally in cases where one or both phases do not have an energy gap, this energy just corresponds to a scale which distinguishes critical fluctuations from those characterizing either phase. Equivalently, $\frac{1}{\Delta}$ is the time scale below which the temporal fluctuations crossover to those characteristic of the critical point. For the transverse field Ising model, the mapping to the classical Ising model implies that the correlation length along the time direction diverges in exactly the same manner as the spatial correlation length and so

$$\Delta \sim \frac{1}{\xi}$$

Thus, the exponent $z = 1$ for the Ising model. In the general case however, no such simple argument is possible for the value of z. The important qualitative point to note is the fact that Δ vanishes as *some* power of ξ^{-1}.

The ordered phase is characterized by a non-vanishing expectation value of the magnetization $m = \frac{1}{N}\sum_i \langle \sigma_i^z \rangle$. On approaching the transition this vanishes with an exponent β which also is the same as that of the classical Ising model in one higher dimension. Similarly the susceptibility to a uniform magnetic field B along the z-direction diverges (from either side) with an exponent γ which again is that of the classical $(d+1)$-dimensional Ising model. Right at the critical point, the magnetization has a power-law dependance on the field B (for small B):

$$m \sim B^{\frac{1}{\delta}}$$

with $\delta = \frac{\gamma+\beta}{\beta}$. In general, for small fields B, and deviations $(r-r_c)$, the magnetization satisfies the universal scaling form

$$m(r - r_c, B) \sim B^{\frac{\beta}{\gamma+\beta}} F\left(\frac{(r - r_c)}{B^{\frac{1}{\gamma+\beta}}}\right) \tag{2.5}$$

Another important quantity characterizing the ordered phase is its stiffness to changes in the boundary conditions. Consider the system that is finite, of linear size L. Now consider the energy cost to change the boundary conditions from periodic to antiperiodic in one of the hypercubic directions. In the ordered phase, this is essentially the cost of creating a domain wall and so scales as L^{d-1}. In the paramagnetic phase, there is a finite correlation length and so this energy cost is exponentially small for large L. Thus the quantity $\Sigma = lim_{L\to\infty} \frac{(E_{anti-per} - E_{per})}{L^{d-1}}$ is finite in the ordered phase but is zero in the paramagnetic phase. On approaching the transition, Σ vanishes with some exponent ζ. This can be related to ν by the following hyperscaling argument. For large finite L, near the transition, the singular part of the energy density with either periodic or antiperiodic boundary conditions satisfies the hyperscaling relation

$$\frac{E_{per./anti-per}}{L^d} \sim \frac{1}{\xi^{d+z}} \Psi_{per./anti-per}(\frac{L}{\xi})$$

That the exponent in the prefactor is $(d + z)$ instead of d can be seen easily from the mapping to the classical problem in $d + 1$ dimensions, and is a consequence of the need to include dynamic fluctuations even to understand the thermodynamics. Thus the difference

$$E_{anti-per} - E_{per} \sim \frac{L^d}{\xi^{d+z}} \left[\Psi_{anti-per}(\frac{L}{\xi}) - \Psi_{per}(\frac{L}{\xi})\right]$$

In the ordered phase, as $L \to \infty$, the left hand side $\sim \Sigma L^{d-1}$. Requiring that the right side have this dependance on L for large L, we find

$$\Sigma \sim \frac{1}{\xi^{d+z-1}} \tag{2.6}$$

Putting $z = 1$ for the Ising model, we get the exponent equality

$$\zeta = \nu d \tag{2.7}$$

Scaling forms can also be written down for various correlation functions in the vicinity of the transition. Most importantly, the order parameter correlation function in imaginary time, defined in the usual way,

$$G(x,\tau) = \langle T_\tau(\sigma_i^z(x,\tau), \sigma_i^z(0,0))\rangle \tag{2.8}$$

satisfies (for large x, τ)

$$G(x,\tau) = \frac{1}{x^{d+z-2+\eta}} g\left(\frac{x}{\xi}, \tau\Delta\right) \tag{2.9}$$

The exponent η is related to the ones introduced earlier by $\gamma = \nu(2-\eta)$.

As usual, the scaling and universality of the critical properties can be understood from the point of view of the renormalization group(RG). As has already been mentioned, the quantum nature of the problem implies that it is necessary to include both space and time dependant fluctuations to understand even the thermodynamics. (The formulation as a $(d+1)$-dimensional classical problem does this naturally). Thus, we imagine some RG scheme in which we integrate out fluctuations at short length and time scales to get an effective action for the remaining degrees of freedom, and then rescale lengths and times to get back a theory with the same microscopic cutoffs as before. If we parametrize the length rescaling factor by s, *i.e* $x \rightarrow x' = \frac{x}{s}$, then it is natural to rescale time such that $\tau \rightarrow \tau' = \frac{\tau}{s^z}$, reflecting the possible anisotropy between space and time. Under this RG, the critical point is described by an unstable fixed point with non-trivial quantum fluctuations. The deviation $|r - r_c|$ is a relevant perturbation and carries the system into the stable ordered or disordered fixed points. As usual the eigenvalue of this perturbation under the linearized RG transformations determines the exponent ν. Turning on a finite B-field is another relevant perturbation, and the corresponding eigenvalue determines the other independant exponent.

Finite temperature properties of the system in the vicinity of the quantum critical point can be undertood from the following observations. Consider the mapping to the classical $(d+1)$-dimensional problem. A finite temperature in the quantum problem corresponds to a finite size $\beta = \frac{1}{T}$ along the imaginary time direction in the classical problem. Thus finite temperature scaling forms for physical observables

can be inferred from the theory of finite size scaling at classical critical points. In particular, it is clear that a finite temperature is a relevant perturbation with eigenvalue exactly equal to the exponent z. Scaling forms at finite temperature can then be written down using quite standard arguments. These and other interesting details of the finite temperature properties near quantum critical points can be found in the references[9, 8].

2.2 Other models

In this section, we will introduce some other simple models showing quantum phase transitions, and describe some of their properties in the absence of randomness. This will set the stage for a discussion of their random versions in later chapters.

2.2.1 Discrete Symmetry

The Ising model has a discrete symmetry group Z_2. It is natural to consider generalizations where the symmetry group is still discrete, but different from Z_2. Interesting, well-studied examples in the classical context are the q-state Potts and clock models. Here we will consider their quantum versions.

The models are defined in terms of a variable that can assume q possible states (which we denote $|0\rangle$, $|1\rangle$, $\cdots$, $|q-1\rangle$) on the sites of a d-dimensional lattice. The *classical* Potts (clock) interaction in the presence of a uniform external "magnetic" field H along the '0' direction is

$$\mathcal{H}_{P,int} = -J\sum_{\langle i,j\rangle}\delta_{n_i n_j} - 2H\sum_i(\delta_{n_i,0} - \frac{1}{q}) \tag{2.10}$$

$$\mathcal{H}_{C,int} = -J\sum_{\langle i,j\rangle} 2\cos(\frac{2\pi}{q}(n_i - n_j)) - 2H\sum_i \cos\frac{2\pi n_i}{q} \tag{2.11}$$

where P and C stand for Potts and Clock respectively. We introduce quantum fluctuations into these models by adding at each site a "transverse field" term that attempts to change the state of the variable at that site. Thus we consider the

quantum Hamiltonians

$$\mathcal{H}_P = -h\sum_i \left(\sum_{n_i,n_i'=0}^{q-1} \frac{1}{q}|n_i\rangle\langle n_i'| \right) + \mathcal{H}_{P,int} \tag{2.12}$$

$$\mathcal{H}_C = -h\left(\sum_{n_i=0}^{q-1} (|n_i\rangle\langle n_i+1| + h.c) \right) + \mathcal{H}_{C,int} \tag{2.13}$$

(We identify $|n_i+q\rangle = |n_i\rangle$). Note that at $H=0$, the Hamiltonian $\mathcal{H}_P$ is invariant under a global permutation $|n\rangle \rightarrow |n'\rangle$ of the states at each site. For $\mathcal{H}_C$, the symmetry is a global cyclic rotation $|n\rangle \rightarrow |n+1\rangle$. Clearly for $q=2$, both these models reduce to the transverse field Ising model. For general q, just as in the Ising case, the "transverse field" term plays the role of a kinetic energy that opposes the tendency to order due to the interaction term. Also as in the Ising case, the d-dimensional q-state quantum Potts (clock) model Eqn. 2.12 (2.13) at zero temperature may be regarded as equivalent to a $d+1$-dimensional q-state classical Potts (clock) model[12].

The mapping to the classical $d+1$-dimensional pure problem provides a rather complete picture of the possible phases and the transitions between them. For instance (at zero H), the ferromagnetic ($i.e$ $J>0$, $h>0$) quantum Potts chain has a first order transition for $q>4$, and a second order transition for $q \leq 4$ (for which all the exponents are known exactly and depend on the value of q)[32, 13]. The ferromagnetic clock chains, on the other hand, have, for $q>4$, a quasi-long-range ordered (QLRO) phase sandwiched between a truely long-range ordered phase and a disordered phase[11] . For $q \leq 4$, the quasi-long-range ordered phase disappears and is replaced by an ordinary second order phase transition for which again all the exponents are known exactly[32, 11]. In dimension $d>1$, the quantum Potts models, for any $q>2$, have first order transitions. Later we will see that randomness drastically modifies this picture.

In Appendix A, we rewrite the Potts and Clock Hamiltonians in terms of a set of operators R_{ix}, R_{iz} defined at each site of the lattice. This will be quite useful in later chapters to do calculations.

2.2.2 Continuous Symmetry

We now consider a class of models with a continuous symmetry group - the O(N) quantum rotor models. These are defined in terms of an N-component unit vector $\hat{n}_i$ at each site i of a d-dimensional lattice. The Hamiltonian is

$$H = \frac{g}{2} \sum_i L_{i\mu}^2 - J \sum_{<ij>} \hat{n}_i . \hat{n}_j \tag{2.14}$$

where the $L_{i\mu}$ are the μ'th generator of rotations of $\hat{n}_i$, and so satisfy the appropriate commutation relations. The components of $\hat{n}_i$ commute with each other. Operators at two different sites also commute with each other. The first term is the kinetic energy of a single particle moving on the surface of a sphere in N-dimensions, while the second term is a potential energy tending to orient the $\hat{n}$ vectors at two adjacent sites in the same direction. It is easy to see by using the same procedure as in the Ising case that this quantum model at $T = 0$ is equivalent to the classical O(N)-model at its finite temperature.

For large $\frac{g}{J}$, in any dimension, or for any $\frac{g}{J}$, in $d = 1$, the ground state is paramagnetic with a finite excitation gap. If $d > 1$, then for small $\frac{g}{J}$, the ground state spontaneously breaks the O(N) symmetry, and aquires LRO in $\hat{n}$. The lowest lying excitations in this case are linear dispersing spin waves, and hence gapless. These two phases are separated by a second order transition at a critical value of $\frac{g}{J}$. As in the Ising case, the universal properties at both $T = 0$ and $T \neq 0$ in the vicinity of this transition can be obtained from those of the classical O(N) model.

2.3 Discussion

In this chapter, we have outlined the theory of quantum phase transitions in pure systems, using the transverse field Ising model as the primary example. An important distinguishing feature of quantum transitions is the intertwining of the static and dynamic aspects of the critical phenomena. This implies that to understand the critical behaviour, we need to consider fluctuations in both space and time. In the vicinity of the phase transition, there is a cross-over from fluctuations characteristic of either phase to those associated with the critical point at a large length scale ξ and

a large time scale $\frac{1}{\Delta}$. Both these scales diverge at the critical point, and generally

$$\Delta \sim \frac{1}{\xi^z}$$

thereby defining the exponent z. In the special examples that we discussed in ths chapter, $z = 1$, but in general this is not required to be so.

In many cases, it is possible to make progress in understanding a quantum transition by mapping it to the critical point of a classical statistial mechanical model in one higher dimension, the extra dimension reflecting the need to include time-dependant fluctuations in the quantum problem. This classical model could in general be considerably anisotropic with respect to the extra "time" dimension (corresponding to the possibility of $z \neq 1$). Finite temperature in the quantum problem corresponds to a finite size in the "time" direction in the classical problem. Thus finite temperature scaling forms near the quantum transition can be written down using standard ideas from the theory of finite-size scaling near classical critical points.

In the remaining chapters, we will try to see to what extent these ideas developed in the context of quantum transitions in pure systems carry over to systems with quenched randomness. Is the general scaling structure preserved, perhaps with different exponents and scaling functions, or does randomness alter the physics in more fundamental ways? At the time of writing this thesis, there is no general understanding of quantum transitions in random systems, and so we will address this question by studying various specific models.

Chapter 3

Random Quantum Phase Transitions - Generalities

In this chapter, we will discuss some general features of quantum phase transitions in random systems. We start with the example of the transverse field Ising model in the presence of various kinds of randomness, and outline various possible phases. The rest of the chapter is however considerably more general, and not restricted to this model. Then we will present an argument for a general exponent inequality at a random phase transition originally proposed by Harris[14] and later generalized and proved by Chayes et.al[15]. This inequality also provides a criterion for when, weak randomness, regarded as a perturbation at the critical point of a pure system, changes the universality class (*i.e* is a relevant perturbation). We then consider systems for which the transition is first order in the absence of randomness. It can be shown that in low enough dimension any amount of randomness (that does not however change the symmetry of the system) drives the transition second order[16, 17] while in higher dimensions,the system is stable to a weak randomness. We then introduce the notion of Griffiths singularities: these are singularities in the properties of the system that arise due to the effects of rare, anamolously strongly coupled regions of the sample that locally appear to be in the phase opposite to that of the bulk system. Such singularities are ubiquitous in random systems, but are particularly strong in random quantum systems. We end the chapter with a discussion of some of the problems encountered when trying to apply conventional approaches of phase

transition theory to understand quantum transitions in random systems.

3.1 Random transverse field Ising model

There are two fundamentally different kinds of randomness that one can potentially introduce in the transverse field Ising Hamiltonian.

(i) Bond randomness:- Here the couplings J and h of the Hamiltonian 2.1 are random functions of position, but the symmetry of the system as a whole is unchanged:

$$H = -\sum_{<ij>} J_{ij}\sigma_i^z\sigma_j^z - \sum_i h_i\sigma_i^x \tag{3.1}$$

It is easy to see that this is equivalent to a $(d+1)$-dimensional classical Ising model with disorder correlated along the time direction. Depending on the particulars of the distributions of the J's and h's, there are three possible phases at $T = 0$. As before, we have the ferromagnetic and paramagnetic phases (with properties modified due to the randomness). The new possibility is that of the spin-glass. In this phase, the spins are frozen in time in some random spatially-varying pattern that is determined by the particular realization of randomness in each sample. The total magnetization of the sample is zero, but it is distinguished from the paramagnetic phase by the non-vanishing of the so-called Edwards-Anderson order parameter:

$$Q_{EA} = \lim_{\tau\to\infty} \overline{\langle\sigma_i^z(\tau)\sigma_i^z(0)\rangle} \tag{3.2}$$

where the overbar denotes averaging over the disorder, and angular brackets averaging over quantum fluctuations.

In general, there will be non-trivial phase transitions between any two of these three phases.

(ii) Random fields:- This corresponds to adding to each site of the lattice a random magnetic field that couples to the order parameter σ_i^z, *i.e* an extra term:

$$H_{ran-field} = -\sum_i B_i\sigma_i^z \tag{3.3}$$

Clearly we no longer have the Z_2 symmetry of the non-random model. The case of most interest is when the fields B_i are distributed with zero mean. Then the

ferromagnetically ordered phase can be destroyed, at $T = 0$, either by increasing the quantum fluctuations as before, or by increasing the strength of the randomness. In fact, in low enough dimension, arbitrarily weak random fields are sufficient to destroy the ordered phase.

We will study random field disorder in detail in Chapter 6. In the remainder of this chapter, unless otherwise mentioned, we consider only bond disorder.

3.2 Weak randomness

3.2.1 The Harris-Chayes inequality

If the randomness is weak, it is possible to decide whether it changes the properties of the transition in the pure system. For second order transitions, this is given by the Harris-Chayes inequality[14, 15] (whose validity is however not restricted to weak randomness). Here we will derive it heuristically. A rigorous proof may be found in Ref.[15].

Let r be the parameter whose value has to be tuned to place the system at the critical point. r could, for instance, be the strength of quantum fluctuations in a magnetic system, the density of charge carriers near a metal-insulator transition, the magnetic field near a superconductor-insulator transition, etc. We will, for the time being, consider the case where the main effect of the randomness is to cause fluctuations in the local value of r at which the transition occurs. Consider the weakly random system. Let the mean value of r at which the transition occurs be $r_{c,R}$. The local value of r_c in a region of linear size L will fluctuate about $r_{c,R}$ by an amount which, typically, is $\sim \frac{1}{L^{\frac{d}{2}}}$. This fluctuation becomes bigger than the deviation $r - r_{c,R}$ at length scales $L \sim |r - r_{c,R}|^{-\frac{d}{2}}$. For the randomness to be unimportant, we require that this scale be smaller than the correlation length, *i.e*

$$|r - r_{c,R}|^{-\frac{d}{2}} \ll |r - r_{c,R}|^{-\nu}$$

which implies that $\nu d > 2$. Thus weak randomness is a relevant perturbation at the pure fixed point if $\nu d < 2$. In fact, the result of Ref.[15] implies that at any random

phase transition, we have the exact inequality:

$$\nu \geq \frac{2}{d} \tag{3.4}$$

Thus the inequality holds even for transitions which have no pure analog - such as the paramagnet-spin glass transition, percolation, etc.

More generally, consider the case where the randomness couples to some local operator $O(x,\tau)$ which has a scaling dimension ζ_O at the pure fixed point, *i.e*, under a renormalization transformation appropriate for the pure fixed point $x \to x' = \frac{x}{s}$, $\tau \to \tau' = \frac{\tau}{s^z}$,

$$O(x,\tau) \to O'(x',\tau') = s^{\zeta_O} O(x,\tau)$$

Now consider avaraging over the disorder using replicas. This generates a term $\Delta^2 \int d^dx d\tau_1 d\tau_2 \sum_{ab} O_a(x,\tau_1) O_b(x,\tau_2)$ where Δ^2 is the variance of the random coupling constant. The scaling of Δ^2 is given by power counting to be $\Delta'^2 = s^{d+2z-2\zeta_O}\Delta^2$. Thus the randomness is relevant if $d + 2z - 2\zeta_O > 0$. For the case where the randomness couples to the energy density, this reduces to the criterion $\nu < \frac{2}{d}$ given earlier.

3.2.2 Effects of weak (bond) randomness on first order transitions

In the foregoing we assumed that the pure transition was second order. What happens if it is first order? We now discuss this question.

Consider the pure system at its transition point. A first order transition implies that there is phase coexistence, *i.e*, (for a magnetic system), the paramagnetic phase coexists with all the ordered phases. Now consider the random case. In any region of the system which is, say, in the paramagnetic phase, random local fluctuations in the parameters will make it favourable to create a droplet of an ordered phase of linear size L by an energy gain $\sim L^{\frac{d}{2}}$. The energy cost to do this is essentially just the cost to create the surface between the two phases, and so $\sim L^{d-1}$. For large L, it is therefore energetically favorable to create these droplets if $d < 2$, in which case the phase coexistence is destroyed. Hence, for any weak amount of randomness, for

$d < 2$, the first order transition is converted to second order[47, 17]. The rigorous argument by Aizenmann and Wehr[17] in fact shows that this is so even in the marginal case $d = 2$.

If $d > 2$, then for weak enough randomness, the first order transition is stable, but may become second order at stronger randomness.

3.3 Griffiths singularities

In this section, we show, following the results of Ref.[7, 18, 19, 20, 21, 22] that the presence of randomness introduces singularities in various physical properties even away from any critical point. For concreteness, consider the random transverse field Ising model in its paramagnetic phase. Due to the randomness, there would, in general, be a non-zero probability that any given bond is stronger than the critical bond strength at which the system orders as a whole. This would happen in an entire, compact region of linear size L with probability $P(L) \sim e^{-cL^d}$ where c is a constant determined by the microscopic couplings, width of random distribution, etc. Such regions constitute clusters of spins that are coupled strongly enough that if they were infinite in size they would order. Consider any such cluster of size L. For large L, all the spins in the cluster behave coherently in space, and it is legitimate to treat the cluster as a single giant spin in the presence of some effective transverse field h_L. Thus the cluster has two low-lying energy levels with an energy difference $2h_L$ well separated from other higher energy levels. This effective field, and hence the splitting between the two levels goes to zero as $L \to \infty$, thus giving rise to broken symmetry and long-range order within the cluster. For finite L, h_L can be estimated in perturbation theory in the ratio of the transverse field to the bond strength. To zeroth order of perturbation theory, there is no transverse field, and the cluster has two degenerate ground states (all spins up or down) and other excited states separated by a large energy (of order the bond strength). A non-zero transverse field breaks this ground state degeneracy, and there remains instead a doublet with a non-zero but small splitting. It is clear that this effect will appear only in a large order of the perturbation theory (= number of spins in the cluster)

$\sim L^d$. Thus the splitting, and hence h_L are exponentially small in L^d: $h_L \sim he^{-c_1 L^d}$. Now, we assume that different clusters in the system may be treated independantly of one another. Consider the density of low-energy excitations as measured by the disorder-average of the imaginary part of the local dynamic susceptibility:

$$\chi''_L(\omega) = \overline{\sum_\alpha |\langle\alpha|\sigma_i^z|0\rangle|^2 \delta(\omega - (E_\alpha - E_0))} \tag{3.5}$$

where α refer to eigenstates of the system with energy E_α, and 0 is the ground state. For low ω in the paramagnetic phase, the only contribution will be from the rare clusters discussed above. Thus

$$\begin{aligned} \chi''_L(\omega) &\sim \int dL\ P(L)\delta(\omega - 2h_L) \\ &\sim \int dL\ e^{-cL^d}\delta(\omega - he^{-c_1 L^d}) \\ &\sim \frac{1}{\omega^{1-\frac{c}{c_1}}} \frac{1}{(\ln(\frac{1}{\omega})^{\frac{d}{d-1}}} \end{aligned}$$

Therefore we have the striking result that the paramagnetic phase is gapless with a singular power law (up to logarithmic corrections) density of states at low energies. The power is non-universal and could in principle even lead to a divergent density of states at zero energy. This power law singularity leads to singularities in the thermodynamic properties of the system at low temperature.

Note that the entire effect arises due to the presence of statistically rare clusters which are anomalously strongly coupled, and hence is a unique feature of the random system. The effect becomes weaker with increasing dimension, ultimately vanishing in the limit of infinite dimension. We have also assumed, in our discussion, that the interactions are short-ranged. Increasing the range of the interactions also weakens the effect - for infinite range interactions, there are no singularities. Finally, the effect is also weaker for systems with continuous symmetry, such as the quantum rotor models. Arguments similar to the ones we have used above show that the density of states now only has a weak essential singularity at zero energy[19].

As a historical comment, we mention that singularities in the thermodynamics of a random magnetic system in its paramagnetic phase due to the presence of rare, strongly coupled clusters of spins, were first noted by Griffiths[23] in dilute

classical magnets: hence the name Griffiths singularities. Griffiths showed that the magnetization of a classical dilute magnet above its critical temperature but below that of the pure system has, as a function of an external magnetic field B, a weak essential singularity of the form $e^{-\frac{1}{B}}$. The effects on the dynamics were analyzed later[24], and found to be somewhat stronger: The disorder-averaged spin autocorrelation function was argued to decay non-exponentially $\sim e^{-(\ln t)^{\frac{d}{d-1}}}$. As we have seen, the effect is much stronger in the quantum system both in the dynamics and (hence) the statics.

3.4 Problems with conventional approaches

In this section, we discuss some of the problems we run into in trying to use conventional techniques to understand phase transitions in random quantum systems. The mapping to the classical model in a higher dimension, while still providing useful insight, is not of much help as the disorder is correlated along the time direction, and such classical models have not been studied in great detail before. A fruitful way of understanding critical phenomena in the pure models relies on the observation that mean field theory is exact for all spatial dimension larger than a certain critical dimension d_u. It is then possible to calculate deviations from mean field theory as a perturbation expansion in $\epsilon = d_u - d$[25, 26]. In this section, we discuss the possibility of doing the same for the random models. Consider again the simplest case, the random transverse field Ising model near the paramagnet-ferromagnet transition. Let us assume that the randomness is weak, and ask when it is relevant at the pure fixed point. By the Harris criterion this will be so whenever $\nu_{pure} < \frac{2}{d}$. From the mapping to the $(d+1)$-dimensional classical Ising model, we get $\nu_{pure} = \frac{1}{2}$ for $d \geq 3$, and $\approx .632$ for $d = 2$ and 1 for $d = 1$. Thus weak randomness is relevant for all dimension $d < 4$. For $d > 4$, randomness that is sufficiently weak should not change the critical properties from those of the pure system. For $d < 4$, we might hope to expand in $\epsilon = 4 - d$ and find a fixed point close to the Gaussian fixed point. (See Ref.[27, 28]).To do this, it is necessary to adopt a continuum field theoretic description. For the random transverse field Ising model, this may be written down using

the mapping to the $d+1$-dimensional classical model with randomness correlated along the time direction. Thus we write

$$Z = \int \mathcal{D}\phi e^{-\int d\tau \int d^d x \left[(\frac{\partial \phi}{\partial \tau})^2 + (\nabla \phi)^2 + (r_0 + r(x))\phi^2 + u\phi^4 \right]} \tag{3.6}$$

with $r(x)$ a random function of position with probability distribution $P[r(x)] \sim e^{-\int d^d x \frac{r^2(x)}{2\Delta^2}}$. (Randomness in all the other couplings can be shown to be less important, so we will ignore them). We now average over the disorder using the replica technique to get

$$\overline{Z^n} = \int \mathcal{D}\phi_a e^{-\int d\tau \int d^d x \left[\sum_a (\frac{\partial \phi_a}{\partial \tau})^2 + (\nabla \phi_a)^2 + r_0 \phi_a^2 + u\phi_a^4 \right] + \frac{\Delta^2}{2} \int d^d x \int d\tau d\tau' \sum_{a,a'} \phi_a^2(x,\tau)\phi_{a'}^2(x,\tau')} \tag{3.7}$$

The renormalization group analysis of this action is quite straightforward, so we omit the details. By power counting, we get the tree-level flow equations:

$$\begin{aligned} \frac{dr_0}{dl} &= 2r_0 \\ \frac{du}{dl} &= (3-d)u \\ \frac{d\Delta^2}{dl} &= (4-d)\Delta^2 \end{aligned}$$

Thus Δ^2 becomes relevant below 4 dimensions, consistent with the Harris criterion. Note that the interaction strength u however remains irrelevant upto $d = 3$. At 1-loop, these get modified to

$$\frac{dr_0}{dl} = 2r_0 + c_1 u - c_2 \Delta^2 \tag{3.8}$$

$$\frac{du}{dl} = (3-d)u - c_3 u^2 + c_4 u\Delta^2 \tag{3.9}$$

$$\frac{d\Delta^2}{dl} = (4-d)\Delta^2 + c_5 \Delta^4 - c_6 u\Delta^2 \tag{3.10}$$

where the c_i are all positive constants. These equations do not allow for a fixed point for small ϵ, instead Δ^2 has runanway flows[1].

Thus the ϵ expansion has failed to be of much use, and we need to develop other techniques. Various attempts in the literature[29, 30] to access the random fixed

[1]Curiously enough a fixed point can be found if we assume that the randomness is correlated along ϵ_τ dimension instead of 1, and perform a double expansion in ϵ and ϵ_τ[28]. The meaning of this calculation is unclear to the author.

point perturbatively from some pure fixed point (in (N, d) space where N is the number of spin components) have been unsuccesful, again because of runaway flows for the strength of the randomness. Thus the fixed point theory presumably has a strong amount of randomness. At the level of the non-interacting theory, one expects that the lowest energy modes will be strongly localized. Physically it is clear then that we cannot ignore the effects of interactions - condensation into a localized state leads to enhancement in interaction effects. It is necessary to include both disorder and interactions in a fundamental way.

3.5 Discussion

In this chapter, we have presented some general ideas on random quantum systems near a $T = 0$ phase transition. The inequality $\nu \geq \frac{2}{d}$ is the only rigorous general result that is known to the author. The Griffiths singularities are unique to the random system, and will play an important role in much of our discussion in later chapters. This domination of low energy properties by some statistically rare regions of the sample is ubiquitous in random quantum systems. Whether they affect the properties of the critical point, or whether they are overwhelmed by other physics is an important and, in general as yet unresolved, question. The failure of conventional techniques such as the ϵ expansion is discouraging; it would clearly be worthwhile to have some concrete results even if restricted to special situations. In the next chapter, we will consider models in $d = 1$ and show, following the pioneering work of Fisher[7] on the random transverse field Ising chain, that it is possible to obtain exactly the critical properties of a number of models.

Chapter 4

Results in $d = 1$

The discussion in the previous chapter raised the important question on the role, if any, played by the anomolous effects of rare regions of the sample in determining the properties of the critical point. Conventional techniques such as the ϵ or large-N expansions fail to shed much light on this issue. A considerably more fruitful approach is to examine models in $d = 1$. This is particularly attractive as the effects of the rare regions are expected to be strongest in $d = 1$. Also one-dimensional models are often analytically tractable, so one might hope to obtain reliable results. In this chapter, we will consider several random quantum models with discrete symmetry and obtain exactly their properties near the critical point.

Critical properties of the $d = 1$ random transverse field Ising Chain(RTFIC) have been studied by a number of authors[5, 6, 7, 31]. In particular, Fisher[7] used a real space renormalization technique to obtain a wealth of information and insight on the thermodynamics and static correlation functions. As we will see later, these results, essentially the only reliable analytical calculations in finite dimension, show that the critical properties are quite unlike those at conventional pure quantum transitions. Thus it becomes important to understand if this unusual behaviour is just an artifact of this particular model or if such behaviour is shared by other models. To this end, we will consider the strongly random quantum q-state Potts and Clock models in $d = 1$, and show that the techniques used to solve the RTFIC can also be used to obtain their exact critical properties for any value of q. Remarkably, there is no q dependance in any of the exponents or the scaling functions. This is in stark contrast

to the pure problem where the properties of the transition depend crucially on the value of q[32, 11](See Chapter 2). In addition, considerations on the effects of weak randomness suggest the possibility that for any amount of randomness, the Potts model for any q and the clock model for $2 \leq q \leq 4$ are described by the strong randomness fixed point. For the clock chain with $q > 4$, we suggest the existence of a multicritical point at a finite strength of randomness. Finally, we consider random. quantum versions of the N-component classical Ashkin-Teller models[33] and show again that the critical properties are independant of N, unlike the pure models.

Taken together, the results on these random $d = 1$ quantum critical points are extremely unusual as compared to non-random ones. The effects of rare regions are explicitly shown to dominate the properties not just of the disordered phase but also the critical point, and a portion of the ordered phase close to criticality. Thus they provide a definitive answer, at least in $d = 1$, to the question of whether Griffiths effects are relevant to the critical properties.

The outline of the rest of this chapter is as follows: First, we present the real space RG technique, and derive the exact results mentioned before. Our discussion will be general, and include all the q-state Potts and clock models (the special case $q = 2$ is the Ising model). We point out several features that are unusual and discuss them in detail in the next section.

4.1 Dasgupta-Ma-Fisher Renormalization Group

We will consider random one-dimensional versions of the quantum Potts and clock models. It is most convenient to work in terms of the R-operators introduced in Appendix A.

$$\mathcal{H}_P = -\sum_i h_i \sum_{p=0}^{q-1} (R_{iz}^p + h.c) - \sum_i J_{i,i+1} \sum_{p=0}^{q-1} (R_{ix}^p R_{i+1,x}^{\dagger p} + h.c) \quad (4.1)$$

$$\mathcal{H}_C = -\sum_i h_i (R_{iz} + h.c) - \sum_i J_{i,i+1} (R_{iz} R_{i+1,z}^{\dagger} + h.c) \quad (4.2)$$

We will only consider the case in which all the h_i's and $J_{i,i+1}$'s are positive. We will assume that the h_i and $J_{i,i+1}$ are independent random variables drawn from some distributions $P_1(h)$ and $P_2(J)$ respectively. Defining the total magnetization

as $M = \sum_i \langle s_i \rangle$ with $s_i = \delta_{n_i,0} - \frac{1}{q}$ for the Potts model and $s_i = \cos \frac{2\pi n_i}{q}$ for the clock model, it is clear that as the overall relative strength of the h_i's is decreased, there will be a transition from a phase with $M = 0$ to one with $M \neq 0$. We assume that the randomness is strong and follow closely a real-space RG procedure used by Fisher[7] to extract an enormous amount of information on the RTFIM ($q = 2$ case). The basic idea behind this procedure, first used by Daspupta and Ma[34] in their study of the random antiferromagnetic spin chain, is to successively eliminate the strongest coupling $\Omega = max\{h_i, J_{i,i+1}\}$ in the chain and get an effective Hamiltonian for the low energy degrees of freedom. Consider the case when the maximum coupling is a field, say h_i. We first solve for the part of the Hamiltonian involving h_i. For concreteness, consider the q-state Potts model. The on-site Hamiltonian

$$-h_i \sum_{p=0}^{q-1} (R_{iz}^p + h.c)$$

is trivially diagonalized. The eigenstates, are labelled by a quantum number M_i, and are given by

$$|M_i\rangle = \frac{1}{\sqrt{q}} \sum_{n_i} e^{\frac{2\pi i M_i n_i}{q}} |n_i\rangle$$

The ground state has $M_i = 0$ and has energy $E_0 = -2qh_i$. All the other states with $M_i \neq 0$ have energy $E_M = 0$. For large h_i, excitations of the spin at i cost a large energy. It is therefore legitimate to project the remaining Hamiltonian into the space with the state at i constrained to be $|0_i\rangle$, thereby eliminating the site i. We do this in second order perturbation theory. Appendix B gives the details of the calculation. The result is a new effective bond between the sites $i-1$ and $i+1$ of strength $J = (J_{i-1,i} J_{i,i+1})/\kappa h_i$ where κ is $q/2$. We now have a new random quantum Potts chain with one less site, and one less bond.

On the other hand, if the maximum coupling is a bond, say $J_{i,i+1}$, we first solve for the part of the Hamiltonian involving $J_{i,i+1}$. This is just the Potts interaction between the spins at sites i and $i+1$, and is hence diagonal in the $|n_i, n_{i+1}\rangle$ basis. There are q degenerate ground states in which $n_i = n_{i+1}$. These have energy $-2qJ_{i,i+1}$. All other states with $n_i \neq n_{i+1}$ have energy 0. Clearly, we may think of the q degenerate ground states as corresponding to the q states of a single effective Potts degree of freedom with a magnetic moment equal to the sum of the moments

of the individual spins. For large $J_{i,i+1}$, it is legitimate to project the remaining Hamiltonian into the space with the spins at i and $i+1$ constrained to be in the same state. Again, we do this in second order perturbation theory (see Appendix B). The result is that the two sites i and $i+1$ are replaced by a single Potts variable with an effective transverse field of strength $h = (h_i h_{i+1})/\kappa J_{i,i+1}$ where κ is the same as before (as may be expected from a duality which these models can be shown to possess[35]). To this order of perturbation theory, the interaction of this effective site with the neighbouring spins remains unmodified. We now again have a random quantum Potts chain with one less site, and one less bond.

This decimation procedure is the basic renormalization group transformation. The strategy is to iterate this transformation till the maximum remaining coupling is of the order the energy at which we wish to probe the system. Note that no correlations are introduced between any of the couplings by this procedure. Thus the different bonds and fields continue to be independant random variables, though with distributions that are renormalized. It will also be necessary to keep track of the lengths (in units of the original lattice spacing) of the effective bonds and sites at each stage of the RG. The renormalization transformation for the bond and site lengths is trivial: If we are eliminating a site, say site 2 of length l_{S2} and J_{12}, J_{23} are bonds of length l_{B12} and l_{B23} respectively, then the new bond J_{13} has length $l_{B13} = l_{B12} + l_{B23} + l_{S2}$. Similarly, if we are eliminating a bond, say J_{23} of length l_{B23} between sites 2 and 3 of length l_{S2} and l_{S3} respectively, then the new site 23 has length $l_{S23} = l_{S2} + l_{S3} + l_{B23}$. The bond (site) length l_B (l_S) will be correlated with it's strength J (it's transverse field strength h) but not with any other variables. Thus, it is necessary to consider the renormalization of two joint distributions-that of bond strength and length and that of transverse field strength and site length at any scale Ω.

For the q-state Clock model the same recursion relations are found but with $\kappa = (1 - \cos(\frac{2\pi}{q}))/(1 + \delta_{q,2})$. Below we will analyze these recursion relations for general κ, so they will apply to both the Potts and Clock models.

It is convenient to convert these recursion relations into flow equations for the distributions. From the form of the recursion relations, it is clear that it is natural to

work in terms of the logarithmic variables: $x = \ln(\Omega/J), y = \ln(\Omega/h), \Gamma = \ln(\Omega_I/\Omega)$ where Ω_I is the maximum coupling in the initial distributions, and Ω is the maximum at any given stage of the RG. Denoting the distribution of x and the corresponding bond length l at scale Γ by $P(x, l; \Gamma)$ and that of y and the corresponding site length l at scale Γ by $R(y, l; \Gamma)$, we find the flow equations (the derivation is given in Appendix B)

$$\begin{aligned}\frac{\partial P}{\partial \Gamma} &= \frac{\partial P}{\partial x} + \int dl' P(0, l', \Gamma) \left(\int_0^\infty dl \left[P(0, l; \Gamma) - R(0, l; \Gamma) \right] \right) + \\ &\int_{l_1, l_2, l_3, x_1, x_2} R(0, l_3; \Gamma) P(x_1, l_1; \Gamma) P(x_2, l_2; \Gamma) \\ &\delta(l - l_1 - l_2 - l_3) \delta(x - x_1 - x_2 - x_3 - \ln \kappa) \end{aligned} \tag{4.3}$$

and similarly with $R \leftrightarrow P$. The fact that the flow equations for R and P go over into one another under interchange is implied by the duality symmetry mentioned earlier.

These equations for the case $q = 2$, *i.e*, $\kappa = 1$ have been analyzed in great detail by Fisher[7]. It is, in fact, possible to find an exact scaling solution in the vicinity of the critical point which is *universal* in the sense that the form of the distributions is independant of the choice of initial distributions (except for some pathological ones). Here, we will merely present and discuss the results. We will also argue that the universal scaling solutions for $\kappa \neq 1$ are *identical* to those for $\kappa = 1$.

From the structure of the recursion relations, it is clear that the new coupling generated after each decimation is much smaller than the ones eliminated. Thus the RG transformation pushes the weight of the probability distribution to low values of the bond and field strengths. If the initial distribution is such that all the bonds are bigger than the fields, then it is clear that asymptotically the system renormalizes to a chain with vanishing fields - hence the ground state is ordered. The properties of this phase are qualitatively the same as in the ordered phase of the corresponding pure problem. Similarly in the other limit, when all the fields are bigger than the bonds, the ground state is disordered and qualitatively similar to the pure disordered ground state. However, as we will see later, near the critical point, both phases are substantially different from their pure counterparts due to the presence of Griffiths effects. At the critical point, the RG leads to distributions

which get broader and broader as the maximum energy scale is lowered. Thus, the approximation of perturbatively eliminating the maximum coupling gets better and better. As a consequence, it is believed that the universal results obtainable by this technique in the vicinity of the critical point are in fact exact[7].

4.1.1 Scaling Solution near the critical point

Let δ be a dimensionless parameter measuring the deviation of the system from criticality (see below for a precise definition). For $\kappa = 1$, it is shown in Ref. [7] that for δ small and Γ large (but arbitrary $\delta\Gamma$), the distributions P and R obey the scaling forms

$$P(x, l; \Gamma; \delta) = \frac{1}{\Gamma^3}\mathcal{P}(\frac{x}{\Gamma}, \frac{l}{\Gamma^2}, \delta\Gamma) \tag{4.4}$$

$$R(y, l : \Gamma; \delta) = \frac{1}{\Gamma^3}\mathcal{R}(\frac{y}{\Gamma}, \frac{l}{\Gamma^2}, \delta\Gamma) \tag{4.5}$$

These were found by rescaling the dependant variables and looking for self-dual fixed point solutions, which should describe the critical point. Linearizing the flow equations in the vicinity of the fixed point yields a single relevant perturbation whose strength is parametrized by δ [1]. First consider the simpler situation where we integrate over the bond (site) lengths and look at the distribution of x (y): $\tilde{P}(x; \Gamma; \delta) \equiv \int dl P(x, l; \Gamma)$ and $\tilde{R}(y; \Gamma; \delta) \equiv \int dl R(y, l; \Gamma)$ Right at the critical point, these distributions are found to be simply

$$\tilde{P}_{cr}(x; \Gamma) = \frac{1}{\Gamma}e^{-\frac{x}{\Gamma}} \tag{4.6}$$

$$\tilde{R}_{cr}(y; \Gamma) = \frac{1}{\Gamma}e^{-\frac{y}{\Gamma}} \tag{4.7}$$

This is readily checked by substitution into the flow equations. (The equality of the two distributions is again because of the expected self-duality at the critical point). Thus even in terms of the logarithmic variables x and y, the distributions become extremely broad at low energies (width of the distribution $\sim \Gamma$ which rises indefinitely as we go to lower energies). This implies an extremely broad distribution for the

[1] In Ref[7] for the RTFIM, δ was related exactly to some properties of the original distributions. Such a relation does not seem possible in general.

physical couplings J and h. The critical solution corresponds to the distribution (for h)

$$P_1(h) \sim \frac{1}{h^{1-\frac{1}{\Gamma}}}$$

Note that the power in the denominator approaches 1 as Γ approaches ∞. Thus the distribution is highly singular at the origin - in fact for large enough Γ, the expectation value of $\frac{1}{h}$ will be divergent. It is this extreme broadness of the distribution that validates the entire technique, and also, as we will see later, enables obtaining physical properties of the system with the critical distribution through simple calculations.

The form of the critical distribution Eqns. 4.6 and 4.7 actually implies the scaling of l with Γ^2 in Eqns. 4.4 and 4.5. To see this, consider the fraction n_Ω of the total number of sites that remain at scale Ω. On reducing Ω by an infinetisimal amount $d\Omega$, n_Ω changes because of the elimination of bonds and fields of strength between $\Omega - d\Omega$ and Ω. In terms of the Γ variable, we thus have

$$\frac{dn_\Gamma}{d\Gamma} = -(\tilde{P}(0,\Gamma) + \tilde{R}(0,\Gamma))n_\Gamma$$

The critical point solution Eqn ?? has $\tilde{P}(0,\Gamma) = \tilde{R}(0,\Gamma) = \frac{1}{\Gamma}$. This implies then that at criticality $n_\Gamma \sim \frac{1}{\Gamma^2}$. The inverse of this gives a length scale (the sum of the mean bond and site lengths) $\sim \Gamma^2$.

We can also see now why a non-zero value of $\ln\kappa$ does not change the scaling solution. Imagine looking for scaling solutions to the flow equations 4.3 with x, y, l, Γ going to ∞ but $\frac{x}{\Gamma}, \frac{y}{\Gamma}, \frac{l}{\Gamma^2}$ finite. The argument of the δ-function involving κ in the flow equation for x, for instance, is then changed to $\frac{x}{\Gamma} - \frac{x_1}{\Gamma} - \frac{x_2}{\Gamma} - \frac{\ln\kappa}{\Gamma}$. The last term clearly vanishes in the scaling limit. The value of $\ln\kappa$ thus becomes irrelevant at low energies (so long as it is finite). The resulting probability distributions in the scaling limit described by the fixed point are independent of q. *Thus the value of q merely determines a high-energy cutoff $E_c \sim \Omega_I e^{-|\ln\kappa|}$ (below which one should be in order to observe scaling behaviour) but does not affect the scaling behaviour itself.* Note that since this cutoff goes to zero as $q \to \infty$, our results hold only for finite q.

We now examine some general consequences of the forms of the scaling distributions. The scaling of l with Γ^2 implies it is the logarithm of the energies that scales as

some power of the lengths at this quantum transition. This is in striking contrast to what happens at pure quantum transitions where as we explained in Chapter 2, the energies themselves scale as a power of the length scales. The scaling at this random quantum transition is thus qualitatively different, and can roughly be described as having the exponent $z = \infty$. We will discuss this in detail later.

The scaling forms also imply the existence, away from criticality, of a length scale ξ that diverges at the critical point as

$$\xi \sim \frac{1}{\delta^2} \tag{4.8}$$

This gives the exponent $\nu = 2$, thus saturating the bound provided by the Harris-Chayes inequality. As we will see later, ξ actually sets the scale for the decay of the mean correlation functions. Typical spin correlations will however decay at a different length scale which diverges with exponent 1[6].

In order to calculate the magnetization of the system with and without an external field, it is necessary to keep track of the magnetic moment of the clusters. Each time a bond between two clusters of magnetic moment m_1 and m_2 is eliminated, we get a new cluster of magnetic moment $m = m_1 + m_2$. Thus we are led to consider the joint distribution of cluster sizes, their magnetic moments, and their transverse field strengths (as measured by y) at a scale Γ: $K(y, l, m; \Gamma)$. In the scaling region close to criticality, this can again be shown[7] to satisfy

$$K(y, l, m; \Gamma; \delta) \sim \frac{1}{\Gamma^{3+\phi}} \mathcal{K}(\frac{y}{\Gamma}, \frac{l}{\Gamma^2}, \frac{m}{\Gamma^\phi}, \delta\Gamma) \tag{4.9}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden mean. This scaling distribution is again independant of q.

In the following subsections, we will use these scaling distributions to discuss the scaling of various physical properties of the system. A remarkable feature we will find is that the universal aspects of all the computable critical properties are *independant* of q. We will henceforth refer to this as "superuniversality": These models have different symmetries for different values of q but nevertheless have the same critical properties. Note that this is not necessarily implied by the lack of q dependance in the distributions. The pure problems, after all, trivially have the

same distributions but have very different physical properties. The superuniversality is rather also a consequence of the extreme broadness of the scaling distributions. As we will see, this extreme broadness implies that the critical singularities are determined by the singular form of the distributions rather than by the details of the quantum mechanics of the spin variables in the system.

4.1.2 Thermodynamic Properties

First consider (at $T = 0$), the magnetization $M(B, \delta)$ of the system as a function of external applied field B. In the presence of a magnetic field B, the energy levels of an otherwise-free cluster of magnetic moment m split into a ground state $|n = 0\rangle$ and other excited states with a gap $E_B = 2mqB$ for the Potts case and (atleast) $E_B = 2mB(1 - \cos\frac{2\pi}{q})$ for the clock case. We stop the RG when the maximum coupling $\Omega \sim E_B$. The extreme broadness of the distribution implies that almost all the clusters which have already been eliminated have transverse fields considerably bigger than E_B while almost all that are yet to be eliminated have transverse fields considerably smaller than E_B. Therefore, an asymptotically exact expression for $M(B, \delta)$ is obtained[7] by aligning all the remaining clusters at $\Omega = E_B$ in the direction of the magnetic field. Thus

$$\begin{aligned} M(B,\delta) &= \bar{m} \times (\text{total number of active } (i.e \text{ undecimated}) \\ &\quad \text{spins at scale } \Gamma_B = \ln\frac{D_B}{B}) \\ &\quad + \text{ corrections} \end{aligned}$$

where $\bar{m}$ and D_B are non-universal and possibly q dependent constants. The key point now is that the number of active spins at a given scale Γ is entirely a property of the joint distribution of cluster lengths, magnetic moments, and field strengths which is q-independant in the scaling limit. Consequently, the universal scaling function describing $M(B, \delta)$ will be the same for all q. The q dependance is only in the non-universal quantities $\bar{m}$ and D_B. The scaling function is easily found to be

$$\begin{aligned} M(B,\delta) &= \bar{m}n_{\Gamma_B}\int dy\ dl\ dm\frac{m}{\Gamma^{3+\phi}}\mathcal{K}(\frac{y}{\Gamma},\frac{l}{\Gamma^2},\frac{m}{\Gamma^\phi},\delta\Gamma) \\ &= \bar{m}\Gamma_B^{\phi-2}\mathcal{M}(\delta\Gamma_B) \end{aligned}$$

$$= \frac{\bar{m}}{(\ln \frac{D_B}{B})^{2-\phi}} \mathcal{M}(\delta \ln \frac{D_B}{B}) \tag{4.10}$$

Note that the scaling form is qualitatively different from the corresponding one at the non-random transition Eqn. 2.5. Using standard scaling arguments, we can now obtain the following results: As we approach the transition from the ordered side, the spontaneous magnetization vanishes as $M_0 \sim |\delta|^\beta$ with $\beta = 2 - \phi = \frac{3-\sqrt{5}}{2}$. Right at the critical point, $M_{cr}(B) \sim \frac{1}{(\ln \frac{D_B}{B})^{2-\phi}}$ which is again qualitatively different from the power law behaviour seen at pure quantum transitions.

In Ref. [7], the scaling function $\mathcal{M}$ was calculated exactly. In the limit of small magnetic fields in the disordered phase, $M(B,\delta) \sim B^{2\delta} \ln \frac{1}{B}$ for small δ. Thus the linear susceptibility $\chi = (\frac{dM}{dB})_{B=0}$ is divergent in the disordered phase (at least close to criticality). Note again that this is quite unlike the pure problem. This is a manifestation of the Griffiths effects. The divergence comes entirely from the presence of rare, large clusters with arbitrarily low transverse fields. We will discuss this in detail later.

Similar arguments can be made to obtain the exact scaling forms for the temperature dependance of the susceptibility and the specific heat. All of these are determined (upto non-universal factors) quite simply in terms of the details of the scaling distributions. Thus, there is no q-dependance in any of these scaling functions (apart from in non-universal factors). At the critical point, $\chi_{cr}(T) \sim \frac{1}{T(\ln 1/T)^2}$ while it has power-law T dependance in the ordered and disordered phases. The power is bigger that 1 in the ordered side and smaller than 1 in the disordered side. The divergence of χ at $T = 0$ in the disordered side has already been noted. The power-law divergence in the ordered side is also unusual, and is again a manifestation of Griffiths effects (In the pure problem, $\chi \sim e^{\frac{1}{T}}$ in the ordered side).

4.1.3 Correlation Functions

Consider calculating (equal-time) spin-spin correlations averaged over all realizations of the disorder:

$$\overline{C_{ij}} = \overline{4\langle(q\delta_{n_i,n_j} - 1)\rangle} \qquad (Potts) \tag{4.11}$$

$$= \overline{2\langle \cos(\frac{2\pi}{q}(n_i - n_j))\rangle} \quad (Clock) \tag{4.12}$$

A scaling result for $\overline{C_{ij}}$, in the region of large $|i-j|$, can be found by noting that if spins i and j are both active in the same cluster at some Γ, then their correlator $\sim m^2(\sim o(1))$ and is 0 otherwise. Thus

$$\overline{C_{ij}} \simeq m^2(\text{Prob. that spins } i \text{ and } j \text{ are both active in the same cluster at some } \Gamma)$$

For large $|i-j|$, only large values of Γ are important. This probability then is clearly some property of the details of the particular distribution of bonds and fields which is universal and q-independant in the scaling limit. Thus (upto non-universal factors) the (mean) correlation function is q-independant in the scaling limit. We can again then take over the results of Ref. [7] for the Ising case. $\overline{C_{ij}}$ satisfies the scaling form

$$\overline{C_{ij}} \sim \frac{1}{|i-j|^{\beta}} \mathcal{C}_{\pm}(\frac{|i-j|}{\xi}) \tag{4.13}$$

where $\xi \sim \frac{1}{\delta^2}$ is the correlation length mentioned before. The exponent β in the prefactor follows from standard scaling arguments and the equality $\nu = 2$. The subscripts $+$ and $-$ refer to the ordered and disordered phases respectively. The scaling functions $\mathcal{C}_{\pm}$ are known upto the solution of an ordinary differential equation. In the disordered phase, C_{ij} decays exponentially at length scales beyond ξ. Similarly in the ordered phase, the connected mean correlation function, $i.e\, \overline{C_{ij}}$ with the square of the mean magnetization subtracted out, also decays exponentially at scales beyond ξ. At the critical point $\overline{C_{ij}} \sim \frac{1}{|i-j|^{\beta}}$.

The behaviour of the typical correlation function is somewhat more complicated. For the Ising case, there is considerable evidence[7, 31] that at the critical point it decays as $e^{-c\sqrt{|i-j|}}$. This is presumably also true for the Potts and clock models.

4.2 Weak randomness

The RG procedure used above is valid only if the randomness in the initial distributions P_1 and P_2 is strong. Clearly it cannot address the question whether, if the

initial distribution is narrow, the low energy properties will still be described by the strong randomness fixed point found above. Some insights on this matter are provided by considering the effect of weak randomness on the pure systems. So long as the pure transition is second order, weak randomness is relevant at the fixed point if $\nu < 2/d$ by the Harris-Chayes inequality where d is the spatial dimension. From the known values of ν[32, 13] for $q \leq 4$ for the Potts and clock chains, we conclude that weak randomness is indeed relevant. The simplest scenario then is that for $q \leq 4$, the RG flows take the system to the strong randomness fixed point for any amount of randomness in the initial distributions. The situation however is different for $q > 4$. We discuss the Potts and clock cases separately. The pure Potts chain for $q > 4$ has a first order transition. The result mentioned in the previous chapter on the effect of weak randomness on first order transitions then suggests that the random $q > 4$ quantum Potts chain has a second order transition for any amount of randomness. We conjecture that this transition is described by the strong randomness fixed point.

We now turn to the clock model for $q > 4$ where there is an intermediate quasi-long-range ordered phase separating a truely ordered phase from the paramagnetic phase. The phase transitions into the QLRO phase from either side are of the Kosterlitz-Thouless type with $\nu = \infty$, therefore implying by the Harris-Chayes inequality that weak randomness is irrelevant. The QLRO phase is described by a line of fixed points. Let the fixed point action for a given point on this line be S^*. Now consider adding a little bit of randomness to the system. The randomness only couples to local operators which are invariant under the Z_q symmetry (known as the "energy operator"). The extra term in the action because of the randomness may be written

$$-\int d^d x d\tau \alpha(x) E(x,\tau)$$

with $\alpha(x)$ being a random variable. Now consider averaging over the randomness using replicas. For simplicity, assume $\alpha(x)$ has a gaussian distribution of zero mean and width Δ. The replicated action then reads

$$S_r = \sum_a S_a^* + \Delta^2 \int d^d x d\tau_1 d\tau_2 \sum_{ab} E_a(x,\tau_1) E_b(x,\tau_2)$$

where a, b are replica indices. Now consider performing the renormalization trans-

formation that leaves S^* invariant. Under this x and τ are rescaled such that

$$\begin{aligned} x \rightarrow x' &= \frac{x}{s} \\ \tau \rightarrow \tau' &= \frac{\tau}{s} \end{aligned}$$

Now assume that under this RG

$$E_a(x,\tau) \rightarrow E'_a(x',\tau') = s^{\zeta_E} E_a(x,\tau)$$

(This assumption is, in general, not correct. The correct procedure is to decompose the operator E_a into a sum of "scaling operators" O_{ia} each one of which transforms into itself under the RG: $O'_{ia} = s^{\zeta_i} O_{ia}$. However for our purposes it is sufficient to consider only the operator with the smallest value of ζ_i which we write as ζ_E). By the usual power-counting arguments, Δ^2 scales as

$$\Delta'^2 = \Delta^2 s^{3-2\zeta_E}$$

Thus weak randomness is relevant/irrelevant according to whether $\zeta_E < 3/2$ or not. Now consider the fixed point theory. Through out the QLRO phase, this is described by a Gaussian action for a single scalar field θ (see Ref.[11] for instance) which physically corresponds to the angular orientation of the spins. (The restriction that θ at each site can have only q values is irrelevant at long length scales, as is the constraint that $0 \leq \theta < 2\pi$)

$$S^* = \int d^d x d\tau \frac{K}{2} [(\frac{\partial \theta}{\partial \tau})^2 + (\nabla \theta)^2]$$

The term in the square brackets is itself an "energy operator" and has $\zeta = 2$. Thus randomness in its coefficients is irrelevant. All other "energy operators" are themselves irrelevant at the fixed point and hence have $\zeta < 2$ - randomness in their coefficients is hence even more irrelevant. We conclude that weak disorder is irrelevant through out the QLRO phase.

Thus weak disorder does not change the nature of the phase diagram in this case while for strong disorder, as we have seen earlier, there is a single second order phase transition (surrounded by Griffiths regions) and no QLRO phase. Understanding how the phase diagram changes as the strength of the disorder increases is an interesting open question. We speculate that as the strength of the disorder is increased,

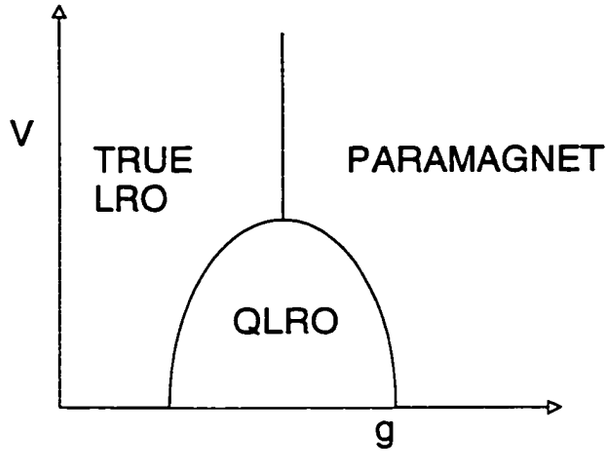


Figure 4.1: One possible (schematic) phase diagram for the $q > 4$ random quantum clock chain. $g = \overline{\ln h}$ is a parameter that measures the strength of quantum fluctuations and V is a measure of the strength of the randomness.

the two lines of Kosterlitz-Thouless transitions bounding the QLRO phase merge at a multicritical point. Beyond this point the QLRO phase disappears, and there is a single second order transition described by the strong randomness fixed point (See Figure 4.1). We do not however have any strong arguments ruling out more complicated scenarios (such as for instance, a new intermediate phase separating the QLRO phase from the region with a single second order transition).

4.3 Ashkin-Teller models

In this section, we consider the critical properties of a class of models - the Ashkin-Teller models - in the presence of strong randomness, and show (at least under certain conditions) that they are again identical to those of the RTFIC. These models are defined by the Hamiltonian

$$H = -\sum_{i\alpha} J_i \sigma^x_{i\alpha}\sigma^x_{i+1\alpha} - \sum_{i\alpha} h_i \sigma^y_{i\alpha} - \sum_{i\alpha\alpha'} K_i \sigma^x_{i\alpha}\sigma^x_{i+1\alpha}\sigma^x_{i\alpha'}\sigma^x_{i+1\alpha'} - \sum_{i\alpha\alpha'} R_i \sigma^y_{i\alpha}\sigma^y_{i\alpha'} \quad (4.14)$$

where the index α runs from 1 to N. Thus the Hamiltonian represents a system of N Ising chains coupled in a certain way. We will assume that all the couplings

are always positive, and also only consider weak interchain coupling K_i, R_i. The properties of the pure models are well-known. The case $N = 1$ is just the Ising model. The case $N = 2$ has a line of fixed points describing the critical point for weak interchain coupling. For $N > 2$, the models have first order transitions. Now consider the random case. Weak randomness is clearly relevant for $N = 1$ and $N > 2$. In fact it is relevant for $N = 2$ as well. This result requires some work, but is true for the following reason. It is well-known that the $N = 2$ Hamiltonian is equivalent to the quantum spin-$\frac{1}{2}XXZ$ chain. The effects of weak randomness at weak $S^z - S^z$ interaction (which corresponds to weak interchain coupling) have been considered by Doty and Fisher[36], and were found to be a relevant perturbation. Thus we expect that the critical properties for all N are controlled by strongly random fixed points. In Appendix B, we use the Dasgupta-Ma-Fisher RG to show that the presence of a small (K_i, h_i) is an irrelevant coupling to the properties of N decoupled strongly random Ising chains. Thus the random Ashkin-Teller models have the same critical behaviour as the Ising model at least at weak interchain coupling - further evidence for the notion of superuniversality.

4.4 Discussion

In this chapter, we examined the critical properties of a number of random quantum models with discrete symmetry in $d = 1$. The properties of all these $d = 1$ random quantum critical points turned out to be extremely unusual as compared to pure quantum critical points - thus it is not just a specific feature of the RTFIC[5, 6, 7]. Below we will discuss the most interesting of these unusual properties.

4.4.1 Activated Dynamic Scaling

As noted earlier, at all these random critical points, it is the logarithm of the energies that scale as some power of the lengths. This is quite unlike pure quantum critical points, where the energies themselves scale as a power of the lengths. This unusual dynamic scaling is similar to what is expected near the critical point of certain *classical* random systems - such as the random field Ising model - which are

controlled by fixed points at $T = 0$. In those systems, the logarithm of the relaxation times diverges as a power of the correlation length - a behaviour known as activated dynamic scaling. The random quantum critical points discussed above can also be thought of as being controlled by fixed points with no quantum fluctuations, as can be seen from the following discussion.

Consider the calculation of any of the singular critical properties. For concreteness, let us take the case of the field dependant magnetization at $T = 0$. We stop the RG at a scale $\Gamma_B \sim \ln \frac{D_B}{B}$. Clusters which have been eliminated before this scale are ignored while clusters which are yet to be eliminated are assumed to be aligned completely in the direction of the field. Thus the entire magnetization scaling function is determined by treating the low-energy clusters classically. In terms of the original spins, each spin contributes a magnetic moment 0 (if it is part of an already eliminated cluster) or a moment of $o(1)$ if it has not yet been decimated. Similarly, in the calculation of the susceptibility or the specific heat at finite temperature, we stop the RG at a scale $\Gamma_T \sim \ln \frac{D_T}{T}$ and treat the remaining clusters as independant, classical spins to get the exact scaling results. Thus, in general, if we imagine renormalizing the system down to a low-energy scale $\Omega \sim e^{\Gamma}$, and wrote down a Hamiltonan for the remaining degrees of freedom which correctly reproduces the critical singularities, then this Hamiltonian can be taken to be classical with an accuracy which gets better with decreasing Ω. In particular, the leading scaling results are exactly obtained from this classical Hamiltonian. The details of this classical Hamiltonian (such as the number of clusters, their magnetic moments, their spatial distribution, etc) are determined by the details of the renormalized probability distributions. Quantum fluctuations do play a role in determining these probability distributions, and hence the classical Hamiltonian. Thus quantum effects are important in determining the low energy Hamiltonian which nevertheless is itself classical. In RG terminology, quantum effects may be considered "dangerously irrelevant" at the fixed point controlling the transition. Activated dynamic scaling is perhaps to be expected then by analogy with the finite temperature phase transitions in classical random systems that are controlled by zero temperature fixed points with dangerously irrelevant thermal fluctuations.

4.4.2 Griffiths Effects

These $d = 1$ results also provide an explicit example of Griffiths effects in random quantum systems. In the vicinity of the critical point, both the ordered and disordered phases are *gapless* though there is a finite length scale over which spatial correlations decay. This is again to be contrasted with the pure systems where there is a finite gap away from the critical point in either phase. As explained in the previous chapter, this difference is due to the presence of rare, large clusters of spins which locally appear to be in the opposite phase. In the disordered Griffiths phase, these are some rare clusters which have anamolously low transverse fields. In the ordered Griffiths phase, these are segments of the chain connected together by anamolously low bond strengths. We will now discuss these Griffiths effects in the ordered and disordered phases and the critical point in turn.

(i) Disordered Griffiths Phase:- Here at low energies most of the transverse fields are much bigger than the bonds. Then, a simple model for the thermodynamic properties of the disordered Griffiths phase is as a collection of disconnected clusters with a distribution of transverse field strengths given by

$$P_1(h;\Omega) \sim (\frac{1}{h})^{1-\alpha}\theta(\Omega - h)$$

Near the critical point $\alpha \sim 2\delta$. The presence of clusters with arbitrarily small values of h is responsible for the singular properties of this phase. The power-law singularity in $P_1(h)$ at $h = 0$ gives rise to a power-law density of states (or more precisely the imaginary part of the local dynamic susceptibility) near zero frequency.

(ii) Ordered Griffiths Phase:- This is the dual of the disordered Griffiths phase. At low energies in this phase, most of the bonds are much bigger than the transverse fields. It is natural to model this phase as a classical chain with bond strengths distributed (by duality) according to

$$P_2(J;\Omega) \sim (\frac{1}{J})^{1-\alpha}\theta(\Omega - J)$$

with again $\alpha \sim 2\delta$ on approaching the critical point. It is easily seen that this classical chain with a singular distribution has singular thermodynamic properties. A particularly interesting property is that of the stiffness (see Chapter 2). In this

classical model, the stiffness is simply proportional to the weakest bond in the system. For a chain of size L, this is easily seen to scale as $L^{-\alpha}$. Thus, the stiffness actually vanishes in the thermodynamic limit though there is long range order and a non-zero spontaneous magnetization.

(iii) Critical point:- We now come to the important question of the role played by Griffiths effects in determining the singularities of the critical point. In these systems, the critical singularities are consequences of the singularities in the probability distributions rather than due to long wavelength collective excitations of the spin variables. As we approach low energies near the critical point, nearly all the spins have been eliminated and do not contribute to physical properties. The spins that do contribute are in the very low-energy tails of the distributions and form a vanishingly small fraction of the total number of spins. Thus in these 1-d systems, the effects of statistically rare events (such as an anamolously low value of a bond or a transverse field) completely dominate the properties of the critical point. Given this difference in the physics behind these critical points and pure ones, it is perhaps not surprising that the properties are qualitatively different.

4.4.3 Superuniversality

A remarkable property of the results in $d = 1$ is that the universal critical properties seem to be, to a large extent, independant of the symmetry of the model (at least at the strongly random fixed points): All the Potts, Clock and Ashkin-Teller models have identical critical exponents and even identical scaling functions for the magnetization and mean correlations. We have referred to this property as superuniversality as it goes well beyond the usual universality observed near critical points. The Ising model is a special case of all these models: The exact results for the Ising model of Ref. [7] can therefore be directly taken over to exactly solve for the critical properties of all these other models. Note that many of the corresponding results are not known exactly for the pure problem. This further emphasizes the non-intuitive fact that it is possible to get more exact information on the random problem than the pure one.

We do not at this stage have a deep understanding of the phenomenon of su-

peruniversality. From the details of the calculations, it is clear that the fact that the critical singularities are properties of the probability distributions is crucial. We have associated this with the dangerous irrelevance of quantum fluctuations at the critical fixed point. Whether this is a sufficient condition for superuniversality is an open question. In particular, in these $d = 1$ models, a deep reason for why the distributions must be independant of the symmetry is lacking.

Finally, a few fringe comments: Our results imply that for classical 2-dimensional Potts and clock models with disorder correlated along one direction, the critical properties of the *finite temperature* phase transition are independent of the value of q. It is interesting that numerical simulations by Chen et. al.[37] on the classical 2-d 8-state Potts model with uncorrelated disorder find a second order transition with exponents equal to the classical 2-d Ising values. This suggests the possibility that the q-independence found here with correlated disorder also holds with uncorrelated disorder. Some evidence for this has also been found recently[38, 39]. However other calculations by expanding in $q - 2$ [40] or in the limit of large q[41] do find q-dependance in the exponents. Further studies to address this issue will be welcome. From our discussion of the phase diagram for the clock chains with $q > 4$ (Figure 4.1), there arises the possibility of a novel multicritical point at finite randomness. Verification of the existence of this point, perhaps numerically, is an interesting open problem.

Chapter 5

Higher Dimensions: Dilute Models Near Percolation Threshold

The critical properties of the $d = 1$ random quantum models discussed in the previous chapter are very unusual as compared to conventional quantum critical points. Specifically, length scales were found to diverge logarithmically with energy scales, implying a dynamic critical exponent $z = \infty$. The transition was shown to be flanked on either side by "Griffiths" regions with a susceptibility diverging due to contributions from statistically rare fluctuations. Further the critical properties were identical for a wide class of models with different symmetries. There are very few reliable results on other random quantum transitions, especially in finite dimensions $d > 1$; thus the next important question to understand is if the anomalous properties of the random quantum chains are a specialty of $d = 1$, or if there exist higher dimensional quantum transitions which share these properties.

A number of features of the $d = 1$ transitions are reminiscent of a percolation transition [42] in the lattice. Consider, for instance, the calculation of the spontaneous magnetization in the ordered side near the critical point. As we have emphasized, each spin either contributes 0 (if it was part of a cluster that was eliminated before a scale $\Gamma_\delta \approx \frac{1}{\delta}$) or of order 1 (if it is "active" in a cluster that has not yet

been eliminated). As we approach the critical point, the contribution of the active spins to the magnetization does not go to zero; rather it is the number of active spins themselves. *The vanishing of the magnetization at the critical point is due to the vanishing of the number of active spins, each of which contributes* $\sim o(1)$, *the rest contributing* 0. Thus if we consider the set of lattice points on which the spins that are active at any energy scale lie, then that set gets increasingly dilute as we go to low energy scales. This is very much like what happens at a percolation transition. Thus in searching for higher dimensional systems which could potentially be similar to the $d = 1$ systems, we are naturally led to study quantum models on dilute lattices in the vicinity of the percolation threshold.

In this chapter, we will study the simple example of the transverse field Ising model on diluted lattices. We will demonstrate a quantum transition in this system, in any higher dimension $d > 1$, that shares many of the unusual properties of the 1-dimensional systems. The transition is controlled by a fixed point with no quantum fluctuations, and displays activated dynamic scaling, superuniversal critical behaviour, and Griffiths regions away from criticality. We will also see that, as in the $d = 1$ case, the critical singularities are entirely due to the Griffiths effects. We are not claiming here that these are generic to any random quantum transition. Our point rather is that these properties are not just a pathology of $d = 1$, and could, in principle, be realized in any finite dimension.

We consider bond or site diluted Ising models with short range interactions. As was first suggested by Harris [14], and as we argue below, these models have two quantum transitions (See Fig 5.1): At low dilution, below the percolation threshold, there is a phase transition when the long range ferromagnetic order is destroyed by increasing the transverse field. This is expected to be in the universality class of the generic random bond quantum Ising transition. Right at the percolation threshold, there is a finite range of transverse field strengths at which the system remains critical. There is thus another quantum transition, across the percolation threshold, at low but non-zero transverse field strengths, which is potentially in a different universality class. These two critical lines meet at a multicritical point (M).

The properties of the second transition, at the percolation threshold, are de-

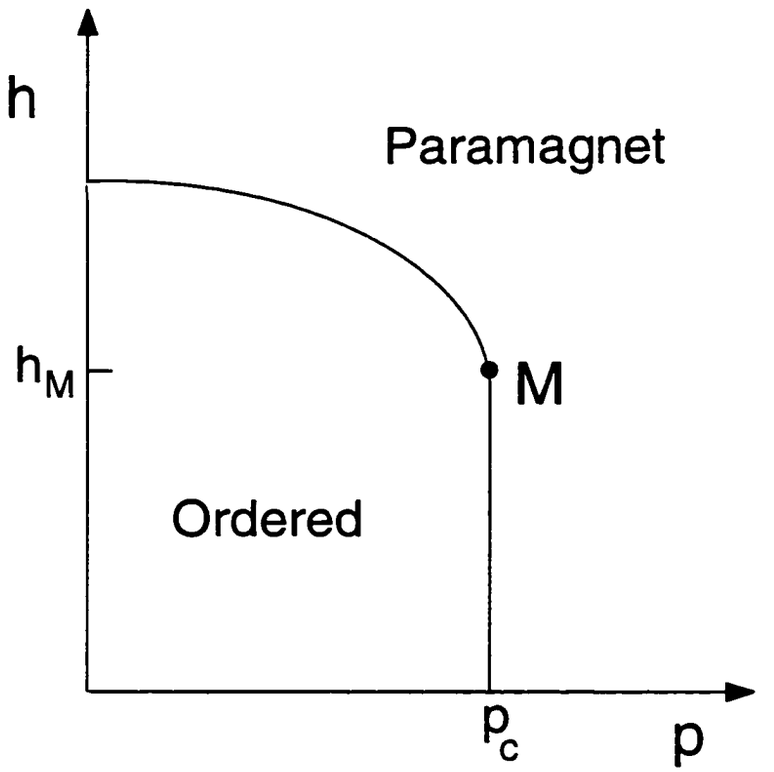


Figure 5.1: Phase diagram of the dilute Ising model in a transverse field (h) at zero temperature. The dilution probability is p. The multicritical point M is at $p = p_c$, $h = h_M$. The quantum transition along the vertical phase boundary ($h < h_M$, $p = p_c$) is controlled by the classical percolation fixed point at $p = p_c$, $h = 0$; quantum effects (due to a non-zero h) are dangerously irrelevant, and lead to activated dynamic scaling near the $h < h_M$, $p = p_c$ line.

termined largely by the statistics and geometry of the percolating clusters about which much information is available. This permits us to make definitive statements about the scaling properties of this transition in any dimension. To our knowledge, these are the first reliable analytical results on any random quantum transition in dimension $d > 1$. We show that the dynamic scaling is activated, with length scales diverging logarithmically with energy scales. We also demonstrate the existence of Griffiths phases on either side with diverging susceptibility, and the superuniversality of the critical properties. Our approach shows clearly the connection between these behaviors, and $T = 0$ phase transitions at which quantum fluctuations are "dangerously irrelevant" [45]; indeed, various exponents of the transition are given by those of the classical percolation transition, and are hence known either exactly or numerically. We also obtain bounds on exponents characterizing the multicritical point.

5.1 Classical Dilute Ising Models

Before plunging into a study of dilute quantum models, we briefly review the properties of classical Ising models on a dilute lattice. Consider, for definiteness, the bond-diluted classical Ising model defined by

$$H = -\sum_{<ij>} J_{ij}\sigma_i^z\sigma_j^z \tag{5.1}$$

where $\langle ij \rangle$ labels the nearest-neighbor sites of a d-dimensional lattice ($d > 1$), and J_{ij} equals 0 with probability p and equals $J > 0$, with probability $1-p$. At $p = 0$, as the temperature T is increased, there is a phase transition from an ordered state to a disordered one (See Fig 5.2). On the other axis, when $T = 0$ there is a percolation transition at $p = p_c$.

The geometric properties of diluted lattices both close to and away from the percolation threshold have been studied quite extensively, and are very well-understood. For $p < p_c$, there is a thermodynamically large connected cluster, while for $p > p_c$, there only are finite connected clusters. Right at $p = p_c$, there are a large number of clusters with a broad distribution of sizes. These clusters are known to have a fractal

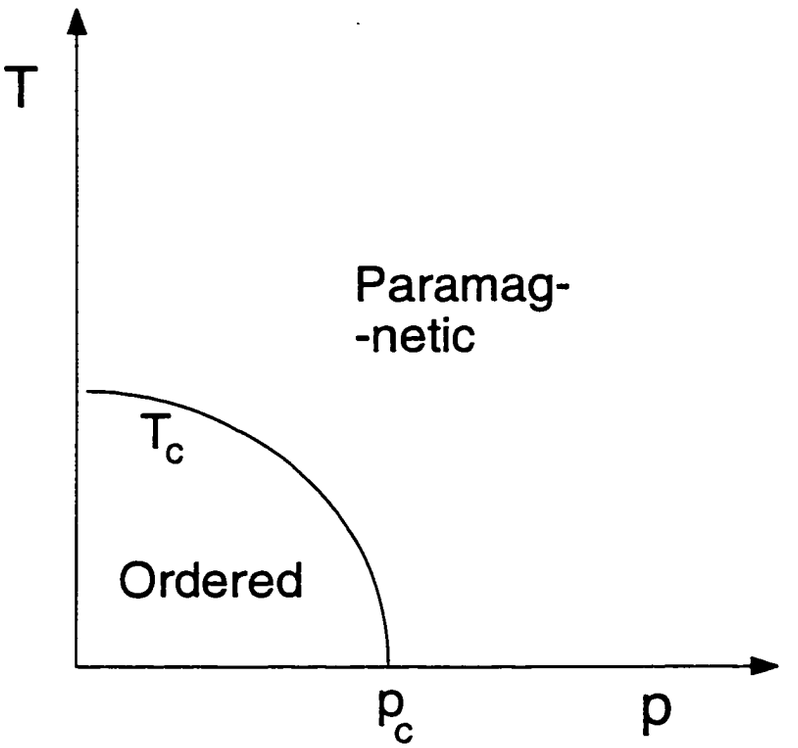


Figure 5.2: Phase diagram of the classical dilute Ising model at finite temperature. The dilution probability is p. The phase boundary goes to zero as $p \rightarrow p_c-$ as $T_c \sim 1/\ln(1/(p_c - p))$.

structure. Though no cluster is thermodynamically large, there will be an infinite connected cluster with a fractal dimension less than the true spatial dimension. An important fact about the critical percolating cluster is that it consists of arbitrarily long one-dimensional segments which are crucial for it's connectedness. Breaking these segments splits the cluster into two disjoint units.

For $p < p_c$, for small enough T, the system retains long range order. This is ultimately destroyed for large enough T. On the other hand, if $p > p_c$, there is no long range order for any T. The effects of a small finite temperature for $p \approx p_c$ have been analyzed by a number of authors[44]. It was found that finite T is always a relevant perturbation at $p = p_c$. For Ising systems, T_c goes to zero as $p \rightarrow p_c-$

as $T_c \sim \ln \frac{1}{p_c - p}$. These results can be understood in the following way. As we mentioned earlier, the critical percolating cluster consists of a number of arbitrarily large one-dimensional segments. These segments are the weakest links in the cluster; correlations in the cluster will be destroyed if they are destroyed along the segments. From the low temperature behaviour of the $d = 1$ (classical) Ising model, we know that any finite temperature will destroy correlations in a large segment over a length scale ξ_T that is exponentially large in $\frac{1}{T}$. In RG language, finite T is a (marginally) relevant perturbation at the $T = 0$ fixed point of the $d = 1$ classical Ising model, hence finite T is relevant at the percolation threshold as well. Now consider the infinite cluster at $p < p_c$. This resembles the critical clusters at scales $\ll \xi$, where $\xi \sim (p_c - p)^{-\nu_p}$ is the percolation correlation length. At larger scales, there is a crossover to the geometry of a d-dimensional lattice. Thus thermal effects will destroy correlations in this cluster when $\xi_T \sim \xi$ which leads to $T_c \sim \ln \frac{1}{p_c - p}$.

It is believed that for all $p < p_c$, the transition is in the same universality class governed by a fixed point which is nevertheless different from the percolation fixed point as well as the non-random fixed point (whenever demanded by the Harris-Chayes inequality as happens below $d = 4$). We will however not discuss this further here.

5.2 Quantum Dilute Ising Models

5.2.1 Phase diagram

For concreteness, we again consider *bond*-diluted Ising models defined by the Hamiltonian

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} \sigma_i^z \sigma_j^z - \sum_i h \sigma_i^x \tag{5.2}$$

Unless otherwise mentioned, we will consider only the $T = 0$ properties of the system. The transverse field h is taken to be non-random, although our results remain valid for weakly random distributions as well. At $p = 0$, as h is increased, there is a zero temperature phase transition from an ordered ground state to a disordered one (See Fig 5.1). On the other axis, when $h = 0$, so that the system is classical, there

is a percolation transition at $p = p_c$. For $p < p_c$, for small enough h, the system retains long range order. This is ultimately destroyed for some $h > h_c(p)$, with $h_c(p)$ expected to be a monotonically decreasing function of p. On the other hand, if $p > p_c$, there is no long range order for any h.

Now consider $p = p_c$. Again there is zero magnetization and no long range order for any h. However the system stays critical for $h < h_M = h_c(p_c)$ (Fig 5.1). To see this, note that, although there is no thermodynamically large connected cluster at p_c, there still is an infinite connected cluster with a fractal dimension smaller than the spatial dimension. The spins on this cluster align together at $h = 0$. A small but non-zero h is not sufficient to destroy this order on the critical cluster. That this is true can be seen by the following argument: The critical cluster is definitely more strongly connected than a one-dimensional chain of Ising spins. Even in $d = 1$ where fluctuation effects are strongest, a small h preserves long-range order in the Ising spins. Thus a small h will certainly preserve the order in the critical cluster. Note that the effects of quantum fluctuations are thus quite different from the effects of thermal fluctuations that we discussed in the previous section. The root of this difference lies in the observation that *while any amount of thermal fluctuations destroy the order in the $d = 1$ Ising chain, it takes a finite strength of quantum fluctuations to do so.* In fact, two spins on any sufficiently large finite cluster remain strongly correlated with each other in space for small h. (Of course, for a finite cluster there will be no long-range correlation in time). The critical cluster eventually loses order when h reaches h_M.

To make this more precise consider the disorder averaged, equal time (τ), two point spin correlation function. Spins at points 0 and x are correlated only if they belong to the same cluster; thus $G(x) \equiv [\langle \sigma^z(x, \tau = 0)\sigma^z(0,0)\rangle - \langle \sigma^z(x,0)\rangle\langle \sigma^z(0,0)\rangle] = \int dC\, C\mathcal{P}(0 \text{ and } x \text{ belong to the same cluster})\, \mathcal{P}(\text{if } 0 \text{ and } x \text{ belong to the same cluster, they have correlation } C)$, where angular brackets are averages over quantum/thermal fluctuations, square brackets represent disorder averages, and $\mathcal{P}(E)$ is the probability of the event E. At $p = p_c$, the first probability $\sim x^{-d+2-\eta_p}$ for large x [42]. The arguments above imply that the integral over C is non-zero and independent of x for large x for $h < h_M$. Thus $G(x, p = p_c) \sim x^{-d+2-\eta_p}$ for large x for $h < h_M$, and

there is a critical line for $h < h_M$ at $p = p_c$.

The $T = 0$ phase diagram is as shown in Figure 5.1. Note the striking difference with the finite T phase diagram in the classical model (Fig. 5.2).

5.2.2 Phase Transitions

We will primarily focus attention on the transition across this critical line at $p = p_c$. We show that at $T = 0$, the properties of this transition are strikingly similar to the $d = 1$ results for the random quantum chains. First consider the equal-time two point function away from p_c. Arguing as in the previous paragraph, we get $G(x) \sim \mathcal{P}(0 \text{ and } x \text{ belong to the same cluster}) \sim x^{-d+2-\eta_p} f(x/\xi)$ for large x where the correlation length $\xi \sim |p - p_c|^{-\nu_p}$. Similarly, for $p < p_c$ the mean magnetization $[\langle \sigma^z(x,0) \rangle] \sim \mathcal{P}(\text{any given site belongs to infinite cluster}) \simeq (p_c - p)^{\beta_p}$. The exponents β_p, η_p, ν_p are those of classical percolation theory, and are known exactly in $d = 2$ and for $d > 6$. For any $d < 6$, they satisfy $\beta_p = \nu_p(d - 2 + \eta_p)/2$. Note that these calculations are already quite reminiscent of the ones in the $d = 1$ case: Each spin either contributes 0 to the spontaneous magnetization (if it is part of a finite cluster) or of order 1 (if it is part of the infinite cluster). Similarly two spins contribute 0 to the correlation function if they belong to different clusters and of order 1 if they belong to the same cluster.

Now consider dynamic correlations. The imaginary part of the local dynamic susceptibility $[\chi''_L(\omega)] = \Sigma_N \mathcal{P}(\text{given site } i \text{ belongs to cluster of } N \text{ sites}) \int y dy \mathcal{P}(\text{if site } i \text{ belongs to cluster of } N \text{ sites}, \chi''_i(\omega) = y)$. The energy levels of a cluster of N sites can be described for $h \ll J$ as follows: The two lowest levels correspond to the states of a single effective Ising spin with magnetic moment $\sim N$ in an effective transverse field $h_{\text{eff},N}$. For large N, $h_{\text{eff},N}$ can be estimated in N^{th} order perturbation theory to be $\tilde{h} \exp(-cN)$. (In general, there would also be a prefactor that varies as a power of N; we will ignore this as it is subdominant to the exponential in N. When necessary, we will indicate the modifications induced by keeping this prefactor). The quantities $\tilde{h}$ and c are of order h and $\ln(J/h)$ respectively but their precise values depend on the particular cluster being considered. As the distribution of $\tilde{h}$ and c

is not expected to become very broad near the transition,[1] we will replace them by their typical values h_0 and c_0 respectively. Apart from these two lowest levels, there are other levels separated from these by energies $\sim J$. These can be ignored for the low energy physics, and for small $\omega \ll h$, we only need to consider large clusters.

We now need the following results[42] from percolation theory for the probability $P(N,p)$ that a given site belongs to a large cluster of N sites. At $p = p_c$, $P(N,p_c) \sim N^{1-\tau}$ where τ equals $5/2$ for $d > 6$, and equals $1 + d/D$ for $d < 6$ where D is the fractal dimension of the critical clusters. In particular $\tau = 187/91$ in $d = 2$, and is approximately 2.18 in $d = 3$. Away from p_c, for $d < 6$ or $d > 8$, $P(N,p)$ satisfies the scaling form

$$P(N,p) \sim N^{1-\tau} g\left(N/\xi^D\right) \tag{5.3}$$

The scaling function $g(y)$ is universal but is different for $p < p_c$ and $p > p_c$; it approaches 1 for $y \ll 1$, while for $y \gg 1$,

$$\begin{aligned} g(y, p > p_c) &\sim y^{-\theta+\tau} e^{-c_+ y} \\ g(y, p < p_c) &\sim y^{-\theta'+\tau} e^{-c_- y^{1-1/d}} \end{aligned} \tag{5.4}$$

with $\theta = 1$, $3/2$ and $5/2$ for $d = 2, 3$ and $d > 8$ respectively, and $\theta' = 5/4$, $-1/9$ in $d = 2, 3$ respectively. The constants $c_{+,-}$ are of order unity. For $6 < d < 8$, $P(N,p)$ satisfies a more complicated two-variable scaling form [46]. For simplicity, we shall not discuss this case here, though including it is straightforward.

Using these percolation results, we get for the dynamic susceptibility of the Ising model

$$\begin{aligned} [\chi_L''(\omega)] &\sim \int \frac{dN}{N^{\tau-1}} g(N/\xi^D) \delta(\omega - h_0 e^{-c_0 N}) \\ &\sim \frac{1}{\omega(\ln(h_0/\omega))^{\tau-1}} g\left(\frac{\ln(h_0/\omega)}{c_0 \xi^D}\right) \end{aligned} \tag{5.5}$$

[1] In particular, we expect the distribution of c to be bounded. This is because for a given N, the gap should be lowest for a fully d-dimensional cluster and largest for a one dimensional cluster and both of these are exponentially small in N. Even then one may object that taking the distribution of c into account while not changing any of the results qualitatively may modify the details of the scaling functions, etc. While we cannot rule this out definitively, as the distribution of c is correlated with that of N, we think it unlikely. Instead, we conjecture that keeping the distribution of c only leads to corrections to the scaling forms derived below.

Note that the scaling variable is $\ln(1/\omega)/\xi^D$. This is a precise statement of the activated dynamic scaling mentioned earlier, which has thus been shown to occur in all $d > 1$ in the present model. Asymptotic forms in various limits can be obtained from the limiting behavior of $g(x)$ described above. For $p \geq p_c$, we get $\chi''_L \sim \omega^{-1+\alpha}(\ln(h_0/\omega))^{1-\tau}$ with $\alpha \sim \xi^{-D}$ so that at $p = p_c, \alpha = 0$ (Including the power-law prefactor in $h_{\text{eff},N}$ will only change the power of $\ln(1/\omega)$ in by $\sim \xi^{-D}$, and similarly for the prefactor in the expression for $p < p_c$ given below). Note that just on the disordered side, the system is gapless with a power-law density of states. The physical origin of this is, as usual, the presence of rare, large clusters with arbitrarily small energy gaps. For $p < p_c$, the presence of the infinite cluster (and the associated long range order) gives rise to a delta function at $\omega = 0$. For $\omega \neq 0$, $\chi''_L(\omega)$ is still determined by contributions from the finite clusters. Proceeding as before, we find $\chi''_L(\omega \neq 0) \sim (1/\omega)(\ln(h_0/\omega))^{1-\tau} \exp\left(-\kappa(\ln(h_0/\omega))^{1-1/d}\right)$ with $\kappa \sim \xi^{-D(1-1/d)}$. Again the system is gapless. The gaplessness of both the ordered and disordered phases in the vicinity of the transition is unlike quantum transitions in pure systems, but is probably generic to many random quantum transitions[18].

The magnetization in response to a uniform external applied magnetic field H along the $\hat{z}$ direction can be calculated similarly. For small $H \ll h$, only large clusters contribute. The magnetization per site of a cluster of size N is that of an Ising spin of magnetic moment N in a transverse field $h_{\text{eff},N}$, and is therefore given by

$$M_N(H) = \frac{NH}{((NH)^2 + h_{\text{eff},N}^2)^{1/2}}$$

Thus the total magnetization per site (after subtracting the regular contribution of the infinite cluster for $p < p_c$) is

$$M(H) - M(H=0) \sim \int dN \frac{1}{N^{\tau-1}} g(N/\xi^D) M_N(H)$$

The singular part therefore has the scaling form

$$M_{\text{sing}}(H) \sim \frac{1}{(\ln(h_0/H))^{\tau-2}} \Phi\left(c\frac{\ln(h_0/H)}{\xi^D}\right) \tag{5.6}$$

with c a non-universal constant, and $\Phi(y)$ a universal function which is related to $g(y)$ by

$$\Phi(y) = \int_1^\infty w^{1-\tau} dw g(wy)$$

Again note that the scaling variable is $\ln(h_0/H)\xi^{-D}$, which is unlike conventional critical behavior, but is similar to the $d = 1$ random quantum chains.

We thus have the following asymptotic forms as $H \to 0$:

$$M_{\rm sing}(H) \sim \begin{cases} (\ln(h_0/H))^{2-\tau} & p = p_c \\ \xi^D (\ln(h_0/H))^{1-\theta} (H/h_0)^{\gamma} & p \geq p_c \\ \xi^{D(1-1/d)} (\ln(h_0/H))^{-\theta'+1-1/d} e^{-(\gamma' \ln(h_0/H))^{1-1/d}} & p \leq p_c \end{cases} \tag{5.7}$$

with $\gamma, \gamma' \sim \xi^{-D}$ (as with the dynamic susceptibility, including the power-law prefactor in $h_{{\rm eff},N}$ changes the power of $\ln(1/H)$ for $p > p_c$, and the prefactor for $p < p_c$ by $\sim \xi^{-D}$). Note that the magnetization rises as a power of H, with a continuously varying exponent which is smaller than 1 in a region of the disordered phase close to the transition. This is again similar to the $d = 1$ result, and gives rise to a divergent linear susceptibility throughout this region. In the ordered side $dM/dH \sim 1/H$ with weak corrections. Thus the linear susceptibility diverges in the ordered side as well.

So far, we have focussed on the $T = 0$ scaling properties, and demonstrated the similarities with the $d = 1$ results. What about the finite T properties ? For the *classical* dilute Ising model at $p = p_c$, the correlation length at finite T behaves as exp(constant/T) . This is essentially due to the presence of one dimensional segments in the critical percolating clusters. For the quantum problem for $h < h_M$, these one dimensional segments would give rise to a thermal correlation length (ξ_T) with a similar exponential dependence on $1/T$, and a prefactor that is a power-law in T; this is the behavior in the "high T" region of Fig 5.3. In contrast, for the $d = 1$ random quantum Ising chain, the correlation length rises as a power of $\ln(1/T)$ at the critical couplings. Away from the critical point, the crossovers are as shown in Fig 5.3. The low T behavior appears when $\xi \sim \xi_T$, or $T \sim \ln^{-1}(1/(|p - p_c|)$. On the disordered side, the low T system is described well as a collection of rigid Ising clusters with effective transverse fields and a size distribution as before; this leads, for instance, to a linear susceptibility $\chi_T \sim T^{-1+\kappa}$ (up to log corrections) with $\kappa \sim \xi^{-D}$. On the ordered side, there is a finite temperature phase transition; as in the classical case, as $p \nearrow p_c$, the transition temperature falls to zero as $T_c \sim \ln^{-1}(1/(p_c - p))$.

We now turn to the special point M (Fig 5.1). Correlations at the point M should decay faster than along the vertical critical line considered above. Thus, if

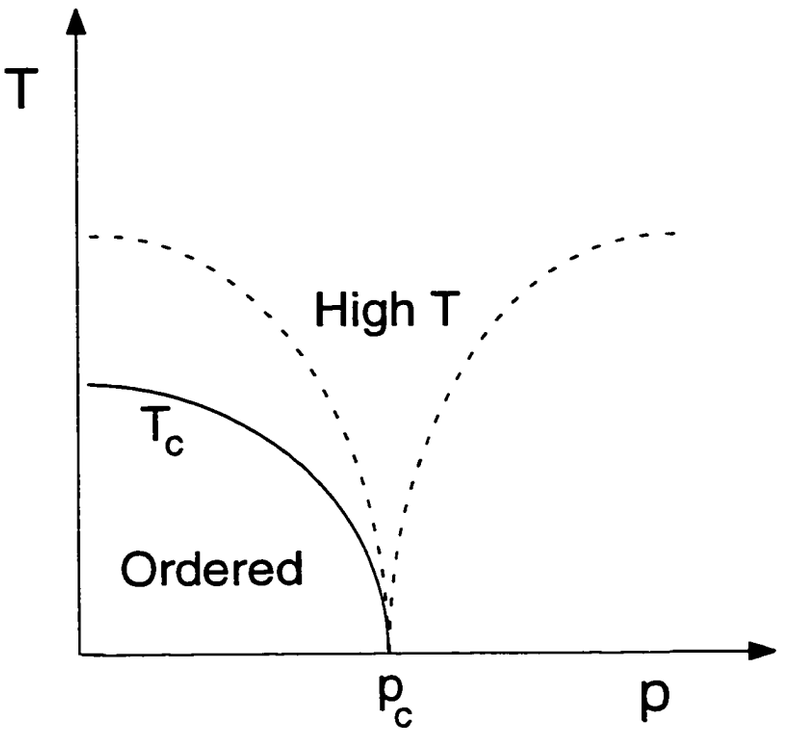


Figure 5.3: Finite temperature phase diagram for $h < h_M$. The dashed lines ($T \sim 1/\ln(1/|p-p_c|)$) represent crossovers from the high T regime, characterized by spin fluctuations on the critical infinite cluster, to the low T regimes. The solid line for $p < p_c$ is the phase transition ($T = T_c \sim 1/\ln(1/(p_c - p))$ where long range order is destroyed.

$G_M(x) \sim x^{-\phi}$, then $\phi > d - 2 + \eta_p$. Similarly, as p approaches p_c from below at h_M, the magnetization should go to zero faster than for $h < h_M$. Thus $\beta_M > \beta_p$. A detailed theory of the scaling at this point we leave as an open question.

5.3 Other dilute quantum models

A further similarity with the $d = 1$ results can be found by generalizing the problem to systems with Potts symmetry. For the diluted models that we have considered here, it should be clear that a vertical critical line exists at the percolation threshold for all q. The exponents and appropriate scaling functions of the corresponding quantum transitions are again independent of q as they are determined mainly by the geometric properties of the lattice near the percolation threshold. As in $d = 1$, all the q dependence is in non-universal quantities and in a high-energy cutoff limiting the regime of universal scaling behavior. Thus superuniversal critical behaviour is again found in every dimension $d > 1$.

Finally it is interesting to ask if this vertical critical line at the percolation threshold continues to exist for systems with continuous symmetry, such as the $O(N)$ rotor models. For $N > 2$, the presence of one dimensional segments in the critical clusters implies rapidly decaying correlations in large clusters. Thus the vertical critical line will not be present for $O(N)$ models for $N > 2$. The $O(2)$ case is special as correlations only decay as a power-law in one dimension. Whether this is like the Ising or the $O(N > 2)$ system we leave as another open question.

5.4 Discussion

To summarize, we have presented a simple example of a random quantum transition in dimensions $d > 1$ which exhibits many of the properties of the transition in the $d = 1$ random quantum chains. In particular, the dynamic scaling was activated, with $\ln(1/\text{energy scale}) \sim \xi^D$. There were Griffiths regions on either side of the transition, with a singular density of states and a diverging susceptibility. Further, the critical properties were superuniversal with all the Potts models having the same

universal critical behaviour. It would be interesting to find experimental systems where this transition can be studied. (One possibility is the system $LiHo_xY_{1-x}F_4$ where, however, the presence of dipolar interactions may complicate the situation.)

Theoretically, an important feature of this transition, as in the $d = 1$ system, is that it is controlled by a classical fixed point with quantum fluctuations being "dangerously irrelevant". The fixed point Hamiltonian in these higher dimensional models is easily identified to be the classical Ising Hamiltonian on a lattice at its percolation threshold. Neverthess, the physical properties of this fixed point Hamiltonian are somewhat pathological and it is necessary to keep the irrelevant quantum fluctuations to get sensible results - hence quantum effects are "dangerously irrelevant". To see this more clearly, consider again the calculation of the field dependant magnetization at the critical point. At the fixed point, all the spins in the system align for any strength of the external field and the magnetization per spin would be 1 for any (positive) value of the field. This is however not correct. Quantum fluctuations prevent spins belonging to small clusters from contributing anything to the magnetization. Spins belonging to large clusters however contribute an amount of order 1. The crossover occurs for a cluster size $N_H \sim \ln \frac{1}{H}$. The leading scaling result is obtained by aligning all clusters with size bigger than N_H. Similarly for the dynamics, the spin autocorrelation function is clearly just 1 at all times at the fixed point as it is classical. (Hence, we may call such fluctuationless fixed points as static). Again this is not correct, and we need to include the irrelevant quantum fluctuations to get the results we presented earlier.

Are these strange critical properties of the $d = 1$ models and the models studied in this chapter necessary whenever the critical fixed point is static? We cannot yet give a definitive answer to this question. However, in the next chapter, we will provide supporting evidence by studying quantum Ising models with field randomness (see Chapter 3, Section 3.1).

Chapter 6

Random Field Quantum Ising Systems

In this chapter, we briefly study the effect of a random longitudinal field applied to the transverse field Ising model. The effects of random longitudinal fields on *classical* Ising models have been studied extensively[47, 49], and are partially well understood. In contrast, very little is known about the corresponding quantum problem in realistic dimensions. Most previous studies of the phase transition in this system have been limited to mean field theory (expected to be valid above 6 spatial dimensions; see below) or to an expansion in $\epsilon = 6 - d$. However, as is well known from the classical problem, these results are not expected to directly be of much relevance to physical systems in finite dimensions much smaller than 6. Nevertheless they give useful evidence to motivate general scaling hypotheses valid in any spatial dimension which we will provide here.

The model in consideration is defined by the Hamiltonian

$$\mathcal{H} = -J \sum_{i,j} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x - \sum_i H_i \sigma_i^z \tag{6.1}$$

For simplicity, we have assumed that J and h are non-random, though our results should apply also for weakly random J and h. H_i is assumed to be random with zero mean and variance Δ^2. The $\vec{\sigma}_i$ are Pauli spin matrices. We note that, as usual, this quantum model in d dimensions at $T = 0$ is equivalent to a *classical* Ising model in $d + 1$ dimensions in a random field with the randomness correlated along one

direction.

It will often be more convenient to consider a coarse-grained continuum field theoretic version of the Hamiltonian 6.1. The continuum action is readily written down as:

$$S = \int d\tau \int d^d x \frac{1}{t} \left[(\frac{\partial \phi}{\partial \tau})^2 + (\nabla \phi)^2 + r\phi^2 + u\phi^4 - H(x)\phi(x,\tau) \right] \qquad (6.2)$$

where $H(x)$ is the (coarse-grained) random field and is taken to be Gaussian distributed with mean 0 and variance Δ^2. For future convenience, we have introduced a factor t as an overall scale for the action.

Later in this chapter, we use the standard Imry-Ma argument to show that the ordered phase is unstable to any weak random field for spatial dimension $d \leq 2$. In Fig 6.1, we show the expected phase diagram for $d > 2$ in the temperature (T), h, Δ space. In what follows, we will concentrate primarily on the $T = 0$ transition from the ordered phase to the paramagnetic phase. Our central claim is that, as in the examples studied in the previous chapters, the quantum transition is controlled by a fixed point with no quantum fluctuations in any dimension (> 2). However quantum fluctuations are "dangerously irrelevant" at this fixed point, and need to be included to get the correct critical behaviour of many physical observables. This claim is completely analogous to the corresponding claim on the role of thermal fluctuations at the finite temperature phase transition in *classical* random field Ising magnets[53]. Indeed, we claim that the *same fixed point* controls both the classical (finite T) and quantum ($T = 0$) transitions in this model. This enables us to relate many of the critical properties of the quantum transition to those of the well-studied (but only partially understood) classical transition. In particular, we suggest that the dynamical scaling is activated, *i.e*, the logarithm of the diverging time scale rises as a power of the diverging length scale.

6.1 Weak randomness

In this section, we will consider the effects of weak random fields on the properties of the pure system. These are quite innocuous deep in the paramagnetic phase when

the transverse field wins over both the randomness and the exchange interaction. So we will turn immediately to the ordered phase and the critical point.

It is easy to see using the Imry-Ma[47] argument[1] that for dimension $d > 2$, the ordered phase of the quantum model is stable to a weakly random H_i. Imagine that in the absence of randomness, the system is deep in the ordered phase. The stability to a weak random field is determined by balancing the energy cost to form large-sized domains with the typical energy gain because of the random H_i. Deep in the ordered phase, the energy cost of a domain $\sim L^{d-1}$ while the typical energy gain due to the random field $\sim L^{d/2}$ exactly as in the classical problem. Thus for $d > 2$, for weak enough random fields, it is not favourable to form large-sized domains and the system stays ordered. We expect of course that strong randomness will destroy the ordered phase in any dimension. Precisely in $d = 2$, as in the classical problem, the system is marginally unstable and there is no ordered phase.

Now, assume that we are at the critical point of the pure system in $d > 2$. When are weak random fields a relevant perturbation at this critical fixed point? This can be answered as follows. Consider the continuum action Eqn. 6.2. Averaging over the disorder using replicas gives the term $\frac{\Delta^2}{2} \int d^d x \int d\tau_1 d\tau_2 \sum_{a,b} \phi_a(x,\tau_1)\phi_b(x,\tau_2)$. Now under the renormaliation group transformation appropriate to the critical fixed point of the pure system $x \rightarrow x' = \frac{x}{s}$, $\tau \rightarrow \tau' = \frac{\tau}{s^z}$, and $\phi \rightarrow \phi' = \phi s^{\frac{d+z-2+\eta}{2}}$. This gives $\Delta^{2\prime} = \Delta^2 s^{z+2-\eta}$. Clearly then Δ is always relevant.

[1]For *classical* random field Ising systems, the question of the lower critical dimension was the subject of a long-standing controversy as field theoretic arguments seemed to predict that any weak random field destroys LRO for all dimensions less than three in disagreement with the Imry-Ma argument. This was finally resolved, in favor of the Imry-Ma prediction, by a rigorous proof by Imbrie[48] that the $d = 3$ classical random field Ising model is ordered for sufficiently weak randomness.

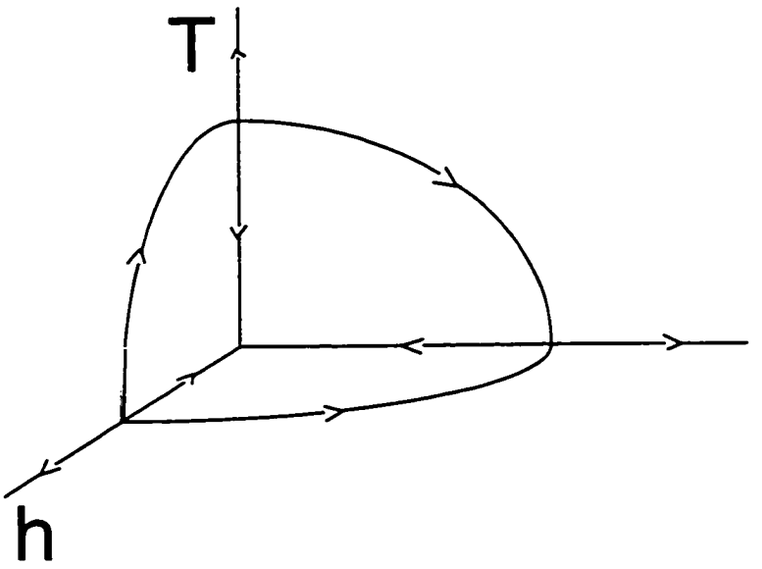


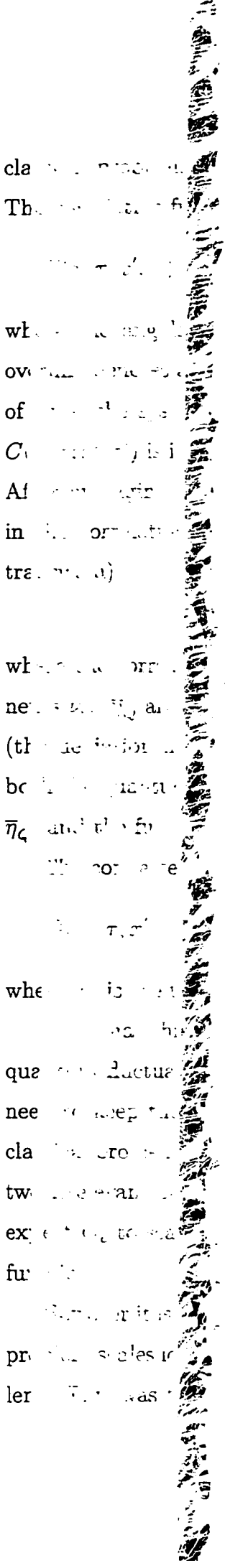
Figure 6.1: Schematic phase diagram and renormalization group flows for the transverse field Ising model in a random longitudinal field for $d > 2$ in the temperature(T), transverse field(h), strength of randomness(Δ) space. There is a transition from a ferromagnet to a paramagnet as any of these three parameters is increased.

6.2 The critical fixed point

In this section, we will present the reasoning behind our assumption that the fixed point controlling the transition has no quantum fluctuations[2].

For *classical* random field magnets, it is believed[49] that the properties of the finite temperature phase transition (which exists for $d > 2$) are controlled by a zero temperature fixed point. The first evidence for this belief came from perturbative studies[47, 50] of a continuum field theoretic description of the magnet near the critical point. Order by order in perturbation theory, it can be shown that the effects of the fluctuations introduced by the randomness dominate over the effects of the ther-

[2]We are always also assuming, perhaps without justification, that the transition remains second order in the presence of the random field



classical problem, there are two different correlation functions that can be defined: The correlation function

$$C(x,\tau;x',\tau') = \overline{<\sigma_z(x,\tau)><\sigma_z(x',\tau')>} - \overline{<\sigma_z(x,\tau)>}\,\overline{<\sigma_z(x',\tau')>} \quad (6.3)$$

where the angular brackets denote averaging over quantum fluctuations and the overline denotes averaging over the randomness. As the randomness is independant of time, the system is translationally invariant in time in every sample. Therefore $C(x,\tau;x',\tau')$ is independant of τ,τ' and we will refer to it as $C(x,x')$ from now on. After averaging over the disorder, spatial translational invariance is also restored in the correlation function, and so $C(x,x') = C(x-x')$. This satisfies (near the transition)

$$C(x) \sim \frac{1}{x^{d-2+\overline{\eta}_Q}} \mathcal{C}\left(\frac{x}{\xi}\right) \quad (6.4)$$

where the correlation length $\xi \sim |h-h_c|^{-\nu_Q}$. The scaling function $\mathcal{C}$ and the exponents ν_Q, $\overline{\eta}_Q$ are properties of the fixed point theory and its relevant perturbations (the deviation from the critical randomness strength). As these are the same for both the quantum and classical transitions, we get the result that the exponents ν_Q, $\overline{\eta}_Q$, and the function $\mathcal{C}$ are identical to their classical counterparts.

The connected correlation function

$$G_c(x,\tau;x',\tau') = \overline{<T_\tau(\sigma_z(x,\tau)\sigma_z(x',\tau'))> - <\sigma_z(x,\tau)><\sigma_z(x',\tau')>} \quad (6.5)$$

where T_τ is the time-ordering symbol. Clearly $G_c(x,\tau;x',\tau') = G_c(x-x',\tau-\tau')$.

Note that this correlation function vanishes at the fixed point (as there are no quantum fluctuations at the fixed point). Thus to obtain its critical behaviour, we need to keep the irrelevant quantum fluctuations. Similarly for the corresponding classical problem, it is necessary to keep the irrelevant thermal fluctuations. As these two irrelevant perturbations may have quite different effects, in general, we *do not* expect G_c to scale identically to the classical space and time dependant correlation function.

However it is possible to argue that the *static* correlation function of the quantum problem scales identically to the *equal-time* correlation function of the classical problem. This was first noticed within the ϵ expansion by Boyanovsky and Cardy[55],

but as we show below is more generally valid (though other results of the ϵ expansion are not). Consider the susceptibility[3]of the model to an external spatially varying static magnetic field $H_{ext}(x)$. Clearly for both the quantum and classical problems, the scaling of this susceptibility near the transition is just determined by minimizing the classical fixed-point Hamiltonian in the presence of the external field. Hence the static (non-local) susceptibilility scales identically for the classical and quantum problems. Now at any finite $T \neq 0$ in the classical problem, the fluctuation-dissipation theorem implies that the equal-time correlation function is proportional to the susceptibility. Similarly, the static correlation function of the quantum problem is also proportional to its susceptibility. Thus these two correlation functions scale identically. We then have the result

$$\int d\tau G_c(x,\tau) \sim \frac{1}{x^{d-2+\eta}} \tilde{\mathcal{G}}\left(\frac{x}{\xi}\right) \tag{6.6}$$

with η and the scaling function $\tilde{\mathcal{G}}$ being identical to those for the connected equal-time correlation functions of the classical problem. The equal-time correlation function in the quantum problem will however scale differently.

We now turn to dynamical correlations. For the classical transition, Villain and Fisher[53] have presented arguments to show that the dynamical scaling is unconventional, with the logarithm of the characteristic relaxation times scaling as a power of the length scales. Similar unconventional dynamic scaling also occurs in the vicinity of the random *quantum* transitions studied in previous chapters, all of which are controlled by fluctuation-less fixed points. We suggest below that the dynamic scaling is activated at this quantum transition as well.

The argument for the dynamic scaling closely follows that for the classical case. Consider a block of the system of size $\sim \xi$ near the critical system. Ignoring quantum fluctuations, the energy landscape as a function of the total magnetization of the block has been argued to scale as ξ^{θ}[53]. Most blocks would thus have a single deep global minimum at some non-zero value of the magnetization. The energy of

[3]It is in principle possible that the susceptibility to a *uniform* external field is divergent in the vicinity of the transition (as happens in a Griffiths phase) thereby invalidating linear response theory. However we expect that for a generic non-uniform field, the susceptibility will be finite and linear response valid.

this minimum will differ from those of other local mimima by amounts $\sim \xi^{\theta}$. The barriers separating these different local mimima also scale as ξ^{θ}. Effects of quantum fluctuations on such blocks should be rather small, and do not contribute significantly to the dynamics. However, there would be some rare blocks in which there are two minima with an energy separation which is nearly zero. The dynamics at long time scales will be dominated by quantum tunnelling between such minima in these rare blocks. As the barrier between these minima rises as a power of ξ, it is natural that the tunneling time $t_Q \sim \exp(c\xi^{\psi})$ with ψ a new exponent. (This quantum tunneling will be important so long as the energy difference between the two minima is roughly less than $\hbar/t_Q$). Thus the dynamics is activated like in the other random quantum transitions studied in previous chapters.

This therefore motivates the following scaling form for the imaginary part of the q, ω dependant susceptibility (which is the spectral density for the connected correlator introduced above):

$$\chi''(q,\omega) = \xi^{\kappa} f(q\xi, \frac{\ln \frac{1}{\omega t_0}}{\xi^{\psi}}) \tag{6.7}$$

where t_0 is a microscopic time scale. The exponent κ will be related to other exponents below. The real part of the susceptibility may be obtained by the Kramers-Kronig relation:

$$\chi'(q,\omega) = \int \frac{d\omega'}{\pi} \frac{\chi''(q,\omega')}{\omega' - \omega}$$

Following Pytte and Imry[56], we write $y = \frac{\ln \frac{1}{\omega' t_0}}{\xi^{\psi}}$ and $z = \frac{\ln \frac{1}{\omega t_0}}{\xi^{\psi}}$. Then

$$\chi'(q,\omega) = \xi^{\kappa+\psi} \int_{-\infty}^{+\infty} \frac{dy}{\pi} f(q\xi, y) \left[\frac{1}{e^{\xi^{\psi}(y-z)} + 1} - \frac{1}{e^{\xi^{\psi}(y-z)} - 1} \right]$$

In the scaling limit when $\xi \to \infty$, we may approximate

$$\begin{aligned} \frac{1}{e^{\xi^{\psi}(y-z)} + 1} &\approx \theta(z-y) \\ \frac{1}{e^{\xi^{\psi}(y-z)} - 1} &\approx -\theta(z-y) \end{aligned}$$

Thus

$$\chi'(q,z) \sim \frac{2}{\pi} \xi^{\kappa+\psi} \int_{-\infty}^{z} dy f(q\xi, y)$$

Thus we get the following scaling form for the real part:

$$\chi'(q,\omega) \sim \xi^{\kappa+\psi} \tilde{f}(q\xi, \frac{\ln \frac{1}{\omega t_0}}{\xi^{\psi}}) \tag{6.8}$$

with $\tilde{f}(x,z) = \frac{2}{\pi} \int_{-\infty}^{z} dy f(x,y)$ In particular, the *static* susceptibility

$$\chi'(q,0) \sim \xi^{\kappa+\psi} \tilde{f}(q\xi, -\infty)$$

Thus we identify $\kappa = 2 - \eta - \psi$. Note that similar scaling forms apply for the dynamics near the classical transition as well, although with a different value for ψ and a different scaling function $f(x,y)$. Nevertheless, as we argued earlier the function $\tilde{f}(x,-\infty)$ should be the same for the quantum and classical transitions.

6.4 Expansion in $\epsilon = 6 - d$

As shown below, the upper critical dimension of this model is 6. It is natural to try an expansion in powers of $\epsilon = 6 - d$. This was done long back by Aharony, Gefen and Shapir[54] and by Boyanovsky and Cardy[55]. Here, we will review their results and discuss them in the context of the general scaling hypotheses of the previous section.

The ϵ expansion is done using the continuum action Eqn. 6.2. It is instructive to set up the renormalization group so that t is allowed to flow while keeping the strength of the randomness fixed. In particular, the results of Ref. [54, 55] show that t flows to zero at the critical fixed point. When $t = 0$, the partition function is determined by the particular configuration $\phi(x,\tau)$ that minimizes the action. Clearly the minimum action configuration is static. Thus solving the fixed point theory simply corresponds to finding the static configuraion $\phi_{stat}(x)$ that minimizes the potential energy terms in the action. Thus the fixed point theory is entirely classical.

First consider the Gaussian theory with $u = 0$. The correlation functions $C(k,\omega) = \overline{\langle\phi(k,\omega)\rangle\langle\phi(-k,-\omega)\rangle}$ and $G(k,\omega) = \overline{\langle\phi(k,\omega)\phi(-k,-\omega)\rangle - \langle\phi(k,\omega)\rangle\langle\phi(-k,-\omega)\rangle}$ can be easily calculated. The result is:

$$\begin{aligned} C(k,\omega) &= \frac{\Delta^2\delta(\omega)}{k^2+r} \\ G(k,\omega) &= \frac{t}{k^2+\omega^2+r} \end{aligned}$$

The critical point is at $r = 0$. The correlation functions introduced in the previous section are seen to scale as

$$C(x) \sim \frac{1}{x^{d-4}} \mathcal{C}(\frac{x}{\xi}) \tag{6.9}$$

$$G_c(x,\tau) \sim \frac{1}{x^{d-1}} \mathcal{G}(\frac{x}{\xi}) \tag{6.10}$$

with the correlation length $\xi = \frac{1}{\sqrt{r}}$. Thus, in the Gaussian theory, we have $\nu_Q = \frac{1}{2}$, $\eta_Q = 0$, $\overline{\eta_Q} = -2$. Note that for the classical problem also, the Gaussian exponents are $\nu = \frac{1}{2}$, $\eta = 0$, $\overline{\eta} = -2$. This is a trivial illustration of the general point made in the previous section. The dynamic scaling is however conventional in the Gaussian theory.

Now consider a momentum shell renormalization group transformation on the Gaussian action. It is convenient to let t flow and keep the strength Δ of the random field fixed. The flow equations for t and r are readily seen to be

$$\begin{aligned} \frac{dt}{dl} &= -3t \\ \frac{dr}{dl} &= 2r \end{aligned}$$

Thus even in the Gaussian theory, t flows to 0 at the fixed point which is hence fluctuationless. The tree-level flow equation for u at the Gaussian fixed point is just obtained by power-counting and is

$$\frac{du}{dl} = (6-d)u$$

Thus interaction effects are irrelevant above 6 spatial dimensions and the Gaussian theory gives the true critical behaviour. For d below 6, it is possible to construct an expansion in powers of $\epsilon = 6 - d$[54]. To leading order, the one loop RG equations are:

$$\begin{aligned} \frac{dt}{dl} &= -3t \\ \frac{dr}{dl} &= 2r + \frac{3ut}{2} K_d \Lambda^{d-1} (1 - \frac{r}{2\Lambda^2}) + 6u\Delta K_d \Lambda^{d-6} (1 - \frac{3r}{\Lambda^2}) \\ \frac{du}{dl} &= (6-d)u - \frac{9u^2 t}{4} K_d \Lambda^{d-3} - 36u^2 \Delta K_d \Lambda^{d-6} \end{aligned}$$

where Λ is the high-momentum cutoff and $K_d = \frac{S_d}{(2\pi)^d}$. ($S_d$ is the surface area of a unit sphere in d dimensions). Again t flows to 0 at the fixed point. Setting

$t = 0$ in the remaining equations it is clear that there is a non-trivial fixed point at $r* = -\frac{\epsilon\Lambda^2}{12}, u* = \frac{\epsilon}{36\Delta K_6}$. The flows can be linearized around this fixed point and give, for instance, $\nu = \frac{1}{2} + \frac{\epsilon}{12}$ to first order in ϵ. Note that this is the same as for the pure problem in $3 - \epsilon$ dimensions. This is a general feature of the ϵ expansion - all the exponents characterizing static critical properties in d dimensions are the same as the pure problem in $d - 3$ dimensions. This result however is an artifact of the ϵ expansion and is not true in all dimensions.

The fact that t flows to 0 to this order means that the new non-Gaussian fixed point is also fluctuationless. In fact the flow equation for t has been shown to be exact to all orders in ϵ[54, 55]. Thus at least within the ϵ expansion, the fixed point is fluctuationless in any dimension.

It was argued by Boyanovsky and Cardy that all the exponents and scaling functions associated with static critical properties of the quantum transition were the same as for the classical transition. As we have seen in the previous section, this result is true quite generally. However dynamic properties (for instance time-dependant correlation functions) were shown to scale differently. Within the ϵ expansion, they found that the dynamic scaling is conventional at both transitions with $z_Q = 1 + (0.0185)\epsilon^2 + (0.0182)\epsilon^3 + o(\epsilon^4)$ and $z_{cl} = 2z_Q$. Thus, as in the classical case, the ϵ expansion is qualitatively incorrect to describe many aspects of the critical behaviour of the random field quantum system well below the upper critical dimension. Nevertheless the ϵ expansion provides useful evidence for the claim that the fixed point has no quantum fluctuations.

6.5 Discussion

The results of this chapter are admittedly not on as firm a footing as those of earlier chapters. Even classical random field Ising models are still not fully understood, and many fundamental questions remain to be answered. What we have done here is primarily to update the theory of the quantum random field models since the pioneering papers in Ref.[54, 55], to take into account subsequent develoments in the understanding of the classical problem. Our main assumption, motivated by the re-

sults of Ref.[54, 55], was that the quantum transition in any dimension is controlled by a fluctuationless fixed point that also controls the classical finite temperature transition. Some of the consequences of this assumption were then examined. In particular, we suggested that the dynamic scaling is of the activated form, with the length scales depending logarithmically on time scales. We have however not provided calculational evidence for this suggestion. Evidence for the activated dynamic scaling cannot be obtained from the ϵ expansion (or from other perturbative approaches like a $2+\epsilon$ expansion). Even for the classical transition, the only available evidence comes from numerical calculations[57] and agreement with the phenomenology seen in experiments[58]. Here, we have argued that activated dynamics in the quantum case is natural if it occurs in the classical problem. We may therefore regard support for activated dynamics in the classical transition as some sort of support for it happening in the quantum transition as well.

There are some other important questions that are left open. For most of the chapter, we have focused on the critical point. The paramagnetic phase should be interesting to study by itself. In particular, is there a finite gap all the way till the critical point, or is there a Griffiths-phase with gapless excitations in the vicinity of the critical point[4]? Similar questions can also be asked about Griffiths effects for the uniform susceptibility in the paramagnetic phase.

[4]It is possible to show for the quantum version of a special model introduced by Grinstein and Mukamel in $d = 1$ that the spin-autocorelation in imaginary time has a stretched exponential form $e^{-c\tau^{\frac{1}{3}}}$. This implies gapless excitations with an essential singularity at zero energy in the local density of states. For more realistic models however, even in $d = 1$, the autocorrelation presumably decays exponentially. The general situation is unclear.

Chapter 7

Infinite range spin glass models

7.1 Introduction

In previous chapters, we have examined the properties of the phase transition from phases with ferromagnetic long range order to paramagnetic phases induced with increasing strength of quantum fluctuations/randomness. In this chapter, we will turn our attention to quantum transitions from a phase with spin-glass order to a paramagnetic phase. In contrast to classical ferromagnets, even the classical thermodynamics of spin glasses is very poorly understood. There are special limits however in which a considerable amount of progress has been achieved. One such is provided by infinite-range spin glass models. As is well-known, classical SG models display a complicated and rich structure even for infinite-range interactions. In this chapter, we will describe progress in understanding the zero temperature quantum phase transition into a spin glass phase in the models considered earlier. An important feature of such infinite-range models is the absence of Griffiths effects. The paramagnetic phase will turn out to have a ground state with excitations separated by a finite energy gap which vanishes continuously on approaching the transition. As discussed in earlier chapters, this is *not expected* to be the case for models with short-range interactions. Nevertheless, these models, being solvable, provide interesting complementary information to that learned from the 1-d models. The basic idea behind the solutions is quite straightforward[59]. One first performs the disorder average using the replica trick and reduces the problem to an effective single site

problem with a self-consistency condition on the auto-correlation function. At the critical point it is possible to solve the self-consistency condition by making a suitable ansatz for the long-time behaviour of the auto-correlation function. In all the models, one finds that the critical auto-correlation decays as $1/\tau^2$ at large imaginary time τ. In addition, in the paramagnetic side, there is an energy gap that vanishes on approaching the transition with an exponent $1/2$ with logarithmic corrections. This was first done for the Ising and $O(M)$ rotor models by Miller and Huse[60], and by Ye, Sachdev, and Read[19]. The extension to the Potts and Clock cases was done by the author and S.N. Majumdar[61]. Below, as a particular example, we will provide details of the calculation for the Potts models. This of course includes, as a special case, the Ising model. The calculation for the other models is very similar, and so we will not discuss them here.

7.2 An example: The Infinite-Range Quantum Potts Spin Glass

The infinite-range quantum Potts spin glass model is described by the Hamiltonian

$$H = -h \sum_i \sum_{p=0}^{q-1} (R_{iz}^p + (R_{iz}^\dagger)^p) - \sum_{i,j} \frac{J_{ij}}{\sqrt{N}} \sum_{p=0}^{q-1} (R_{ix}^p R_{jx}^{\dagger p} + h.c) \tag{7.1}$$

The J_{ij} are assumed to have a gaussian distribution centered at 0 with width J. We have introduced a factor $\frac{1}{\sqrt{N}}$ in the interaction (N is the total number of sites). This is necessary to get a sensible thermodynamic limit. To see this, note that the effective field seen by each spin for any typical configuration of spins is the sum of $o(N)$ random numbers of either sign and magnitude $\sim \frac{1}{\sqrt{N}}$, and is hence expected to be $o(1)$.

Clearly, in the limit when $h \gg J$, the ground state will be paramagnetic. In the opposite limit, when $J \gg h$, the ground state develops spin glass order where the Potts spin at any given site is frozen in time and points along a definite direction. The directions at different sites are different, and is determined by the particular realization of the random interactions J_{ij}. This phase is distinguished from the para-

magnetic phase by the non-vanishing of the (Edwards-Anderson) order parameter

$$Q_{EA} = \frac{1}{q-1}\sum_{p=1}^{q-1}\lim_{\tau\to\infty}\langle T_\tau\left(R_x^p(\tau)R_x^{\dagger p}(0)\right)\rangle \tag{7.2}$$

Here we are concerned more with the properties of the quantum phase transition between these two phases as the ratio $\frac{h}{J}$ is varied. We will make the important assumption that the transition is second order, and obtain exact results for the critical properties. We will also examine the structure of the paramagnetic phase close to the transition, but will not attempt to analyse the spin-glass phase.

The partition function

$$Z = Tr(e^{-\beta H}) = Tr(e^{-\beta(H_0+H_1)}) = Tr\left(e^{-\beta H_0}T_\tau\left(e^{-\int_0^\beta d\tau H_1(\tau)}\right)\right) \tag{7.3}$$

We have denoted by H_0 the transverse field term and by H_1 the interaction term. The symbol T_τ denotes time ordering and the time dependance of any operator $\hat{O}$ is given by $\hat{O}(\tau) = e^{\tau H_0}\hat{O}(0)e^{-\tau H_0}$

It is convenient to write $H_1 = -\sum_{i,j}\frac{J_{ij}}{\sqrt{N}}B_{ij}$. We perform the disorder average using the replica technique. The result is

$$\overline{Z^n} = Tr\left[e^{-\beta\sum_a H_{0a}}T_\tau\left(e^{\frac{J^2}{2N}\int_0^\beta d\tau\int_0^\beta d\tau'\sum_{ij,ab}B_{ij}^a(\tau)B_{ij}^b(\tau')}\right)\right] \tag{7.4}$$

where $a = 1, ..., n$ is the replica index and the notation is (hopefully) self-explanatory. The term with the product of B's can be simplified to get:

$$\sum_{ij,ab,\tau\tau'}BB = \sum_{ab,pp',\tau\tau'}4\left[|\sum_i R_{ixa}^p(\tau)R_{ixb}^{\dagger p'}(\tau')|^2\right] \tag{7.5}$$

We thus have a term quartic in the R operators which can be decoupled by the Hubbard- Stratonovich transformation to get

$$\overline{Z^n} = \int \mathcal{D}\mathcal{B}_{\tau\tau'}^{ab,pp'}e^{-2NJ^2\sum|\mathcal{B}|^2}Tr\left[e^{-\beta\sum_a H_{0a}}T_\tau\left(e^{2J^2\sum(\mathcal{B}_{\tau\tau'}^{ab,pp'}\sum_i R_{ixa}^p(\tau)R_{ixb}^{\dagger p'}(\tau')+h.c)}\right)\right] \tag{7.6}$$

As $N\to\infty$, the $\mathcal{B}$ integral can be evaluated in the saddle point approximation. At the saddle point, different sites decouple from each other, and the problem reduces to solving a single site quantum problem defined by the partition function

$$Z_0 = Tr\left[e^{\beta\sum_{ap}h(R_{ia}^p+R_{ia}^{\dagger p})}T_\tau\left(e^{2J^2\sum_{ab,pp',\tau\tau'}(\mathcal{B}_{\tau\tau'}^{ab,pp'}R_{ia}^p(\tau)R_{ib}^{\dagger p'}(\tau')+h.c)}\right)\right] \tag{7.7}$$

with the self-consistency condition

$$\mathcal{B}^{ab,pp'}_{\tau\tau'} = \left\langle T_\tau(R^{p'}_{xb}(\tau')R^{\dagger p}_{xa}(\tau))\right\rangle_0 \tag{7.8}$$

We now assume that we are in the paramagnetic phase or at the critical point and so look for replica diagonal, permutation symmetry unbroken saddle point solutions. As shown in AppendixC, this implies

$$\mathcal{B}^{ab,pp'}_{\tau\tau'} = \delta^{ab}\delta^{pp'}\mathcal{B}_{\tau\tau'} \tag{7.9}$$

Then different replicas decouple, and so we drop the replica index from now on. We thus finally get the single site partition function

$$Z_0 = Tr\left[e^{\beta h\sum_p(R^p_z+h.c)}T_\tau\left(e^{2J^2\int_{\tau\tau'}\sum_p(\mathcal{B}_{\tau\tau'}R^p_x(\tau)R^{\dagger p'}_x+h.c)}\right)\right] \tag{7.10}$$

with the self-consistency condition

$$\mathcal{B}_{\tau\tau'} = \left\langle T_\tau(R_x(\tau')R^\dagger_x(\tau)\right\rangle$$

To make further progress, it is convenient to think of this single-site quantum problem at $T = 0$ as a classical 1-dimensional Potts model with an additional (possibly long-ranged) interaction $\mathcal{B}_{\tau,\tau'}$ So consider the classical partition function

$$Z_{cl} = \sum_{n_i} e^{K\sum_i \delta_{n_i,n_{i+1}}+2J^2\sum_{ij}\mathcal{B}_{ij}\delta_{n_i,n_j}} \tag{7.11}$$

where $n_i = 0, ..., q-1$ and K is a positive constant, with the self-consistency condition

$$\mathcal{B}_{ij} = \left\langle \delta_{n_i,n_j}\right\rangle_{cl}$$

At the critical point of the original model, we expect the Potts autocorrelator to decay as a power-law $\mathcal{B}(\tau) \sim \frac{1}{\tau^\alpha}$ at large τ. This implies that the interaction in the corresponding classical Potts model Eqn. 7.11 is long-ranged. It is easy to show that for such a classical Potts chain with a power-law long range interaction, the correlator also decays with the same power through out the high temperature phase. Of course the correlator also has a power law decay at the critical point, but not in any ordered phase. It follows that the self-consistency can only be satisfied if the classical model is in it's high temperature phase or at it's critical point. We will

demonstrate below that it is indeed possible to satisfy the self-consistency with the assumption that it is in the high-T phase and not at it's critical point. So we will henceforth assume that we are in the high-T phase.

It is now convenient to work with a continuum field theoretic formulation of the classical Potts model. This is done[62] in terms of $q-1$ fields ψ_l transforming irreducibly under the action of the group S_q of permutation of q objects.The Landau-Ginzburg-Wilson action describing the Potts model can be written down on general symmetry grounds as:

$$S = \int d\tau \frac{g}{2} \sum_l \left(\frac{d\psi_l}{d\tau}\right)^2 - 2J^2 \int d\tau \int d\tau' \tilde{\mathcal{B}}(\tau-\tau')\psi_l(\tau)\psi_l(\tau') \quad (7.12)$$

$$+ \int d\tau \left(r \sum_l \psi_l^2 + v^{lmn}\psi_l\psi_m\psi_n + u^{lmnk}\psi_l\psi_m\psi_n\psi_k + ... \right) \quad (7.13)$$

with the requirement $\tilde{\mathcal{B}}(\tau) \rightarrow \mathcal{B}(\tau)$ as $\tau \rightarrow \infty$. (Summation over repeated indices is implied). Note that for $q > 2$, there exists a cubic invariant for the group S_q. The coupling constants in this action are expected to be analytic functions of the parameter $h - h_c$ where h_c is the critical transverse field strength at which there is an onset of spin glass order. The Potts correlator is given in terms of the fields ψ_l by $\frac{1}{q-1}\langle \sum_l \psi_l(\tau)\psi_l(0)\rangle$. We now solve the self-consistency condition perturbatively in v and u. We will find that the result of the Gaussian theory is in fact exact at long times. Denoting the gaussian propagator by $\tilde{\mathcal{B}}_0$, we find at zeroth order of perturbation theory

$$\tilde{\mathcal{B}}_0(i\omega) = \frac{1}{q-1}\langle \psi_l(i\omega)\psi_l(-i\omega)\rangle \quad (7.14)$$

$$= \frac{1}{\frac{q}{2}\omega^2 - 2J^2\tilde{\mathcal{B}}_0(i\omega) + r} \quad (7.15)$$

Solving for $\tilde{\mathcal{B}}_0$, and choosing the solution that decays as $\frac{1}{\omega^2}$ at large ω, we get

$$\tilde{\mathcal{B}}_0(i\omega) = \frac{\frac{q}{2}\omega^2 + r - \sqrt{(\frac{q}{2}\omega^2 + r)^2 - 8J^2}}{4J^2} \quad (7.16)$$

$\tilde{\mathcal{B}}_0$ has a singularity at $\omega = 0$ if $r = 2\sqrt{2}J$, of the form

$$\tilde{\mathcal{B}}_0(i\omega) \simeq a + b|\omega| + c\omega^2 + ...$$

Within this Gaussian approximation, we then identify $r = r_c = 2\sqrt{2}J$ as corresponding to the critical point of the original quantum problem. We then have the following result for the decay of the critical imaginary time correlator (within the Gaussian theory):

$$\mathcal{B}_0(\tau) \sim \tilde{\mathcal{B}}_0(\tau) \sim \frac{1}{\tau^2} \tag{7.17}$$

It remains to be shown that this result holds to all orders of perturbation theory. Introducing a self energy $\Sigma_(i\omega)$, the self-consistency can be written in the form

$$\tilde{\mathcal{B}} = \frac{1}{\frac{q}{2}\omega^2 + r - 2J^2\tilde{\mathcal{B}} - \Sigma([\tilde{\mathcal{B}}], i\omega)}$$

which can be solved to give

$$\tilde{\mathcal{B}}(i\omega) = \frac{\frac{q}{2}\omega^2 + r - \Sigma - \sqrt{(\frac{q}{2}\omega^2 + r - \Sigma)^2 - 8J^2}}{4J^2}$$

Consider determining Σ upto second order in perturbation theory. The relevant Feynman diagrams are shown in Fig. 7.1. Diagrams A and B give frequency independant contributions, and so only renormalize r. Clearly analytic parts of Σ do not alter the singular structure of the result of the Gaussian theory. The contribution of diagram C to the non-analytic part of Σ is

$$\begin{aligned}
\Sigma_C(i\omega) &\sim \int d\omega' \tilde{\mathcal{B}}_0(i\omega')\tilde{\mathcal{B}}_0(i(\omega - \omega')) \\
&\sim \int d\tau e^{i\omega\tau}(\tilde{\mathcal{B}}_0(\tau)^2 \\
&\sim \int d\tau e^{i\omega\tau}(\frac{1}{\tau^2})^2 \sim |\omega|^3
\end{aligned}$$

Similarly diagram D gives a non-analytic part $\sim |\omega|^5$. Thus the leading non-analytic part of Σ is $|\omega|^3$. Note however that this is too weak to modify the leading non-analyticity of the gaussian propagator. The effects of higher order terms in the perturbation theory is only to produce changes in the coefficient of the $|\omega|^3$ term or even weaker non-analyticities in Σ. Thus to all orders in perturbation theory, the leading non-analytic part of $\tilde{\mathcal{B}}$ is $|\omega|$, and hence the result

$$\mathcal{B}(\tau) \sim \tilde{\mathcal{B}}(\tau) \sim \frac{1}{\tau^2} \tag{7.18}$$

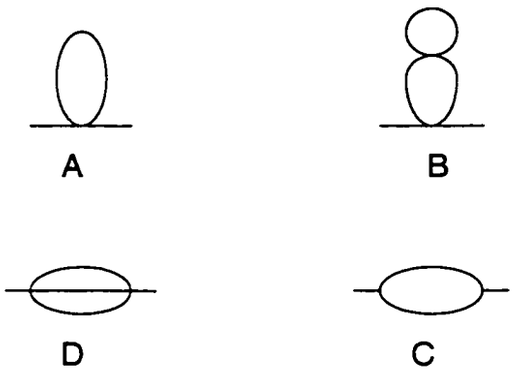


Figure 7.1: Diagrams that contribute to the self-energy to second order for the infinite range Potts spin glass. Diagrams A, B, and D arise due to the quartic interactions while diagram C is due to the cubic interaction.

We now consider the paramagnetic phase close to the critical point. From the arguments given above on the structure of $\Sigma(i\omega)$ at low ω, it is clear that the main effect is to renormalize the values of the constants appearing in the gaussian result Eqn7.16. Thus at low ω, the full propagator has the form

$$\mathcal{B}_{i\omega} = A - B\sqrt{\omega^2 + \Delta^2} + C\omega^2 + ... \tag{7.19}$$

where the ellipses represent terms lower order in ω, Δ. It is easy to see that Δ is the gap to excitations from the ground state. (At the Gaussain level, $\Delta \sim (r - r_c)^{\frac{1}{2}}$; but this will be renormalized by interactions). As we approach the critical point Δ goes to zero, in a manner which can be determined exactly through an ingenious argument first used by Miller and Huse[60] for the Ising case. By definition, it is clear that at equal times, the autocorrelator is exactly equal to 1. This implies that

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathcal{B}(i\omega) = 1 \tag{7.20}$$

The most singular Δ dependance of this integral comes from the square root term in the propagator. This is easily seen to be of the form $\Delta^2 \ln(\frac{1}{\Delta})$. If we assume that the most singular dependance of the propagator on $h - h_c$ is through Δ, then it must be

that the term singular in Δ is cancelled by other terms analytic in $h - h_c$. It follows then that Δ vanishes as $h \to h_c+$ as

$$\Delta \sim (h - h_c)^{\frac{1}{2}} \left(\ln \frac{1}{h - h_c} \right)^{-\frac{1}{2}} \tag{7.21}$$

7.3 Discussion

In this chapter, we have examined the quantum phase transition from a paramagnet to a spin glass phase in models with infinite range interactions, with the q-state Potts model as an example. The infinite range of the interactions destroys the Griffiths effects that definitely exist, and are probably crucial in short-range models. The critical exponents were independant of q, as in the $d = 1$ models considered earlier. We expect that these calculations also describe these models on a hypercubic lattice in the limit of infinite dimensionality. It is natural to ask if one can build on the solutions at $d = \infty$ to construct a theory in finite dimensions. Such an approach was taken in Ref.[19] for the Ising and rotor models. As often happens, it was found that the $d = \infty$ results continued to hold in large enough dimension $d > d_u = 8$. However attempts to reach realistic dimensions by an ϵ expansion have'nt been succesful. It remains to be seen whether the idea of understanding a phase transition by approaching it from high dimension will be fruitful in these problems, or whether the poor treatment of Griffiths effects, inherent in this approach, will limit its utility.

Chapter 8

Summary

In this thesis, we have studied the properties of zero temperature phase transitions in a number of random quantum systems. We showed that fluctuations due to the randomness often played a far more dominant role than quantum effects in determining the critical properties. Thus the physics of the random quantum transitions may be quite different from those of pure systems. This leads to striking differences in the phenomenology associated with random quantum critical points as compared to more conventional ones. In this final chapter, we summarize our results and speculate on their generality and the possible lessons for understanding other random quantum transitions.

(i) Fluctuationless fixed point:-

A principal feature of all the (finite dimensional) quantum transitions we studied was that they were controlled by fixed points with no quantum fluctuations. This is a more formal statement of the point made above regarding the dominant effect of the randomness over quantum fluctuations. As the fixed point theory has no quantum fluctuations, there is no dynamics at the fixed point (hence the terminology "static fixed point"). Thus the dynamics of the transition is determined entirely by the (formally) irrelevant quantum effects. Quantum mechanics is therefore to be considered "dangerously irrelevant" at these fixed points.

It is very unlikely that every random quantum transition is controlled by a static fixed point. But it does appear that such fixed points are somewhat more common in quantum systems as compared to classical systems. The random field systems

studied in Chapter 6 had fluctuationless fixed points in both the classical and quantum cases. But as we have seen, there are a number of other quantum models which are also controlled by static fixed points while their classical analogs are not. For instance, thermal fluctuations are relevant at the percolation threshold of diluted Ising ferromagnets while quantum fluctuations are irrelevant.

There are a number of suggestions in the literature of other quantum transitions with dangerously irrelevant quantum fluctuations. We mention two important examples. First, Kirkpatrick and Belitz[63] have constructed and examined a Landau-Ginzburg theory to describe the metal-insulator transition in disordered, interacting electronic systems. By making an analogy to the classical random field Ising model, they argue that the fixed point of their theory has no quantum fluctuations. The other example is the transition from a metallic spin glass to a metallic quantum paramagnet at $T = 0$. This transition was studied in high dimensions by Sachdev, Read and Opperman[45] and by Sengupta and Georges[64], and was argued to be described by a static fixed point by the former authors. It remains to be seen if this is true in all finite dimensions.

(ii)Activated Dynamic Scaling:-

All the models studied (in finite dimension) also showed very unconventional dynamic scaling, with the logarithm of the diverging time scaling as a power of the diverging length. In contrast, at conventional quantum critical points (in all pure systems, for instance) the diverging time scales as a power of the diverging length, thereby defining the dynamic critical exponent z. Such unconventional dynamics is believed to also occur near certain classical phase transitions in the presence of randomness (such as the random field Ising model) which are controlled by static fixed points. In most of the quantum transitions studied in this thesis, it has been possible to demonstrate both the static nature of the fixed point and the activated form of the dynamics. We speculate that the two phenomena are necessary consequences of each other in low dimensions. The arguments presented in Chapter 6 on the quantum random field Ising model provide some evidence for this speculation. In the Landau theory of the metallic spin glass transition, the dynamics is conventional. Whether this is just an artifact of the Landau theory or whether the dynamics becomes ac-

tivated well below the upper critical dimension (as in the random field problem) is an open question worthy of further investigation.

(iii)Griffiths effects:-

Griffiths phenomena - the domination of physical properties by statistically rare regions ofthe sample- are ubiquitous in random systems. However their effects are rather weak for static thermodynamic properties of classical systems. Somewhat stronger effects are predicted to occur for the dynamics. In quantum systems, the statics and dynamics are inseparable, and it is natural to expect that Griffiths effects are important for static properties as well. In the models of Chapters 4 and 5, we saw explicitly that Griffiths effects dominated the properties of the system in the vicinity of the critical point. In the paramagnetic phase, the susceptibility was divergent in an entire region, and the system was gapless with a power-law density of states at low energy. This is probably generic to many random quantum systems in the vicinity of a critical point (though the effects are less dramatic for continuous symmetry systems). More interestingly, in the models we have studied, Griffiths effects continued to dominate the properties of the critical point itself - the critical singularities were entirely due to the singular properties of rare regions. At this stage, it is unclear how generic this feature is. We speculate that this too is linked with the static nature of the fixed point. If quantum fluctuations are (dangerously) irrelevant, then the effects of randomness dominate and it is perhaps natural that the critical singularities are due to Griffiths effects. As it is unlikely that every random quantum transition is controlled by a static fixed point, it is then also unlikely that the Griffiths singularities determine the critical properties. This is an important issue that future work will hopefully clarify.

(iv) Superuniversal critical behaviour:-

Finally we come to the intriguing property of superuniversality, *i.e* the lack of dependance of the critical properties on the number of spin components, in the models of Chapters 4 and 5. The physics behind this is particularly transparent in the diluted models of Chapter 5. Once again, we do not know how general this is. Here, we mention various other stray results on random quantum systems which are in a similar spirit. First, for the metallic spin glass to the metallic quantum paramagnet

transition mentioned earlier, Sachdev and Read[65] have suggested that the exponents are independant of the number N of spin components of the $O(N)$ quantum rotors. It is noteworthy that this was also suggested to have a static fixed point. Second, if we consider the properties of random quantum $SU(N)$ antiferromagnetic spin chains with the spins in the fundamental representation in one sublattice, and in the antifundamental in the other [1], it can be shown quite straightforwardly using the Dasgupta-Ma-Fisher renormalization group that the low energy properties are (a) universal in the sense that they are independant of most details of the initial distributions, and (b) superuniversal, in the sense that they are completely independant of N. Again, in the absence of randomess, these spin chains have very different properties for different N. Finally, there are interesting numerical results on the Anderson transition in 3 dimension where the critical exponents are found to be very close for the 3 different symmetry classes (orthogonal, unitary and symplectic). It is not clear yet whether they are exactly equal. If that is true, it would be yet another example of some kind of superuniversality.

(v)Other results:-

In addition to the results of this thesis and the work already mentioned in earlier chapters, there have been other studies of random quantum transitions in finite dimensions. Various quantum transitions in random Heisenberg spin chains have been examined [67, 68] using the Dasgupta-Ma-Fisher renormalization group. All of these show activated dynamic scaling and can be thought of as being described by static fixed points. To the author's knowledge, there exists only one calculation of a random quantum transition in $d = 1$ which does not show activated dynamics. This is the transition from a superfluid to an insulator in a system of disordered bosons. This transition was analysed several years ago by Giamarchi and Schulz[69] who found a finite dynamic critical exponent z. While their results may well be correct ultimately, we raise some caveats about their analysis. These authors start from the

[1] Actually these results are closely related to our results on the random quantum Potts chains. Affleck[66] has shown that these $SU(N)$ spin chains are related to the N^2-state Potts models through a Temperley-Lieb mapping. Whether the equivalence holds beyond the Temperley-Lieb mapping is as yet unclear.

superfluid side, and test for the relevance/irrelevance of disorder. The point where disorder becomes relevant is taken to signal the superfluid-insulator transition. This may however be dangerous. If it were to happen that there is an "ordered Griffiths phase" before the true critical point, then disorder would become relevant at the point where the Griffiths phase starts, and not at the critical point. Accessing the true critical point would then not be possible in a weak disorder approach.

There have also been some interesting numerical studies of the paramagnet-spin glass transition in the transverse field Ising model in $d = 2, 3$ dimensions[20, 21]. In both cases, evidence for Griffiths singularities have been found in the divergence of appropriate susceptibilities away from the transition in the paramagnetic phase. However the dynamic scaling at the transition seems conventional.

(vi)Experiments

At the time of writing, there have, unfortunately, been almost no experiments on the kind of simple random quantum statistical mechanical systems that is the subject of this thesis. An interesting exception is the study of the paramagnet-spin glass transition in the diluted dipolar Ising magnet $LiHo_{1-x}Y_xF_4$ in a transverse magnetic field[70]. The critical properties found in the experiment are qualitatively different from those found in the numerical simulations mentioned above, and are probably attributable to the dipolar nature of the coupling in the experimental system.

Exchange coupled Ising magnets will be better candidates as model experimental systems as then the interactions will be short-ranged. However typically exchange interactions are strong, and the strength of the magnetic field required to access the quantum transition rather large. This is at least one difficulty in finding experimental systems with short-ranged coupled Ising spins near the quantum transition. This is however not such a major problem for the transition at the percolation transition studied in Chapter 5. That transition occurs at *small* values of the transverse field (compared to the exchange interaction), and hence may be accessible experimentally. There may of course be other difficulties related to long equillibration times, and being at temperatures low enough to see scaling that may have to be surmounted before the true critical behaviour can be studied.

Appendix A

Potts and Clock models in terms of the R-operators

For use in later chapters, let us introduce at each site i the operators R_{ix} and R_{iz} defined as follows:

$$R_{ix}|n_i\rangle = e^{\frac{2\pi i n_i}{q}}|n_i\rangle \quad (A.1)$$

$$R_{iz}|n_i\rangle = |n_i + 1\rangle \quad (A.2)$$

At each site, R's satisfy

$$\begin{aligned}(R_x)^q &= (R_z)^q = 1\\ R_x R_x^\dagger &= R_z R_z^\dagger = 1\\ R_z R_x &= e^{\frac{-2\pi i}{q}} R_x R_z\\ R_z^\dagger R_x &= e^{\frac{2\pi i}{q}} R_x R_z^\dagger\end{aligned}$$

In terms of these operators, the Potts and Clock Hamiltonians may be written

$$\mathcal{H}_P = -\sum_i h_i \sum_{p=0}^{q-1}(R_{iz}^p + h.c) - \sum_{<ij>} J_{ij} \sum_{p=0}^{q-1}(R_{ix}^p R_{jx}^{\dagger p} + h.c) \quad (A.3)$$

$$\mathcal{H}_C = -\sum_i h_i (R_{iz} + h.c) - \sum_{<ij>} J_{ij}(R_{iz} R_{jz}^\dagger + h.c) \quad (A.4)$$

Appendix B

Recursion relations and flow equations for the $d = 1$ renormalization group

In this appendix, we will provide some details of the calculation of the recursion relations and derive the flow equations.

B.1 Recursion relations

(i) Potts:- Assume a field, say h_2, is the maximum coupling. Consider the 3 site problem involving sites $1, 2$ and 3 and write

$$\begin{aligned} H_0 &= -h_2\hat{A}_2 \\ H_1 &= -(h_1\hat{A}_1 + h_3\hat{A}_3) - J_{12}\hat{B}_{12} - J_{23}\hat{B}_{23} \\ H &= H_0 + H_1 \end{aligned}$$

with $\hat{A}_i = \sum_{p=0}^{q-1}(R_{iz}^p + h.c)$ and $\hat{B}_{i,i+1} = (R_{ix}^p R_{i+1,x}^{\dagger p} + h.c)$. Put site 2 in ground state of H_0 and find the effective coupling between sites 1 and 3 in that subspace by second order perturbation theory. As explained in Chapter 4, the eigenstates of H_0 are $|M_2\rangle = \frac{1}{\sqrt{q}}\sum_{n_2} e^{\frac{2\pi i n_2 M_2}{q}}|n_2\rangle$. The ground state has $M_2 = 0$, and eigenvalue $-2qh_2$. All other states have eigenvalue 0. To second order in perturbation theory,

the required effective Hamiltonian is

$$H_{eff} = \langle M_2 = 0|H_1|M_2 = 0\rangle + \sum_{M_2 \neq 0} \frac{|\langle M_2|H_1|M_2 = 0\rangle|^2}{E_0 - E_{M_2}}$$

The first order contribution is easily seen to be simply $-(h_1\hat{A}_1 + h_3\hat{A}_3)$ (upto additive constants which we will ignore). The second order contribution is

$$\frac{-1}{2h_2 q} \sum_{M_2 \neq 0} |\langle M_2|H_1|M_2 = 0\rangle|^2$$

This is easily evaluated and equals (again upto additive constants)

$$-\frac{2J_{12}J_{23}}{qh_2}\hat{B}_{13}$$

Thus we have

$$H_{eff} = -(h_1\hat{A}_1 + h_3\hat{A}_3) - \frac{2J_{12}J_{23}}{qh_2}\hat{B}_{13}$$

This has the form of a new effective bond between sites 1 and 3 of strength $J_{13} = \frac{2J_{12}J_{23}}{qh_2}$.

Now consider the other case where the maximum coupling is a bond, say, J_{23}. First solve the 2-site problem

$$\begin{aligned} H_{23} &= H_0 + H_1 \\ H_0 &= -J_{23}\hat{B}_{23} \\ H_1 &= -(h_2\hat{A}_2 + h_3\hat{A}_3) \end{aligned}$$

Put $(2,3)$ in ground state of H_0. There are q degenerate ground states with $n_2 = n_3$. Excited states with $n_2 \neq n_3$ are separated from the ground states by a gap $2qJ_{23}$. Labelling the states in the ground state by $n_{23} = n_2 = n_3$, the effective Hamiltonian in the degenerate ground state subspace is

$$\langle n_{23}|H_{eff}|n'_{23}\rangle \simeq \langle n_{23}|H_1|n'_{23}\rangle + \sum_{\alpha \neq gd} \frac{\langle n_{23}|H_1|\alpha\rangle\langle \alpha H_1|n'_{23}\rangle}{E_{gd} - E_\alpha}$$

The calculations are straightforward and the result is

$$H_{eff} = -\frac{2h_2h_3}{qJ_{23}}\hat{A}_{23}$$

where $\hat{A}_{23}$ acts on the effective Potts variable n_{23}. Thus there is a new transverse field $h_{23} = \frac{2h_2h_3}{qJ_{23}}$.

The calculation for the clock model is very similar and leads to the results mentioned in Chapter 4.

(ii) Ashkin-Teller Models:- We now show that small vales of the Ashkin-Teller couplings are irrelevant perturbations on the N decoupled Ising fixed point. If $K_i, R_i \ll J_i, h_i$ to begin with, at least in the initial stages of the RG, only J's or h's will be eliminated. The recursion relations are quite easily derived. If a field h_2 is the maximum coupling, there is a new bond between sites 1 and 3 with

$$\begin{aligned} J_{13} &= \frac{J_{12}J_{23}}{h_2 + (N-1)R_2} \\ K_{13} &= \frac{K_{12}K_{23}}{2h_2 + (N-2)R_2} \end{aligned}$$

Similar relations hold in the other case where the maximum coupling is a bond, the detailed form given exactly by duality. For $R, K \ll h, J$, we have $J_{13} \simeq \frac{J_{12}J_{23}}{h_2}$ and $K_{13} \simeq \frac{K_{12}K_{23}}{2h_2}$ (and similarly for h and R). Thus J and h have the same flows as at zero Ashkin-Teller coupling. Write $u_i = \frac{K_i}{2J_i}$ and $v_i = \frac{R_i}{2h_i}$. u satisfies $u_{13} = u_{12}u_{23}$ (and similarly for v). It is clear that the RG pushes the distribution of u and v to smaller values if they were small to begin with. $\ln u$ and $\ln v$ satisfy simple additive recursion relations exactly like the magnetic moment of a cluster. Thus at scale Γ, $\ln u \sim \ln v \sim -\Gamma^{\phi}$, hence at length scale l, $\ln u \sim \ln v \sim -l^{\phi/2}$. (Note:- The $-$ sign is because of the assumption that initially $u, v < 1$ so that $\ln u, \ln v < 0$.) Thus small values of R, K renormalize very quickly to zero, and we are left with the N decoupled Ising fixed point.

B.2 Flow equations

In this section, we will derive the flow equations Eqn. 4.3 starting from the recursion relations. Let $N(\Gamma)$ be the total number of clusters at scale Γ, and let $\rho_B(x, l, \Gamma), \rho_S(y, l.\Gamma)$ be the total number of bonds of strength $x = \ln \frac{\Omega}{J}$, length l and total number of sites with field strengths $y = \ln \frac{\Omega}{h}$, length l at scale Γ respectively.

Then we have

$$\begin{aligned} P(x,l,\Gamma) &= \frac{1}{N(\Gamma)}\rho_B(x,l,\Gamma) \\ R(y,l,\Gamma) &= \frac{1}{N(\Gamma)}\rho_S(y,l,\Gamma) \end{aligned}$$

Now consider increasing Γ by an infinetisimal amount $d\Gamma$. This involves eliminating bonds with $x \approx 0$ and sites with $y \approx 0$. In terms of Γ instead of Ω, we have $x = \ln \frac{\Omega_l}{J} - \Gamma$. Thus when Γ is changed, x changes by $dx = -d\Gamma$ and similarly for y. Therefore all bonds and sites with $0 < x, y < d\Gamma$ are eliminated which implies

$$N(\Gamma + d\Gamma) = N(\Gamma) - d\Gamma(\rho_B(0,l,\Gamma) + \rho_S(0,l,\Gamma)$$

For the change in $\rho_B(x,l,\Gamma)$, we clearly have

$$\begin{aligned} \rho_B(x,l,\Gamma+d\Gamma) &= \rho_B(x-dx,l,\Gamma) + d\Gamma \int dx_1\, dx_2\, dl_1\, dl_2\, dl_3 \rho_S(0,l_3,\Gamma) P(x_1,l_1,\Gamma) \\ &\quad P(x_2,l_2,\Gamma)[\delta(l-l_1-l_2-l_3)\delta(x-x_1-x_2-\ln\kappa) - \delta(l-l_1)\delta(x-x_1) \\ &\quad -\delta(l-l_2)\delta(x-x_2)] \\ &= \rho_B(x-dx,l,\Gamma) - d\Gamma(2\int dl' \rho(0,l',\Gamma) N(\Gamma) P(x,l.\Gamma) \\ &\quad +d\Gamma \int dx_1\, dx_2\, dl_1\, dl_2\, dl_3 \rho_S(0,l_3,\Gamma) P(x_1,l_1,\Gamma) P(x_2,l_2,\Gamma) \\ &\quad \delta(l-l_1-l_2-l_3)\delta(x-x_1-x_2-\ln\kappa) \end{aligned}$$

Dividing this equation by the one for $N(\Gamma)$, we get the flow equation 4.3. Clearly the equation for R can be derived along the same lines.

Appendix C

Proof of Equation 7.9

In this appendix. we will prove that the permutation symmetry of the Potts model implies that the correlator $\langle T_\tau(R_{ix}^{p'}(\tau)R_{ix}^{\dagger p}(0)\rangle$ in a paramagnetic ground state is independant of p, p' for $p, p' = 1, ..., q-1$. First, by C_q symmetry, it is clear that the correlator is zero unless $p = p'$. Write $R_x^p = O_p$ for $p = 0, .., q-1$. (Drop site index from now on). For $P =$ any member of the permutation group of symmetries of the Hamilitonian,

$$\begin{aligned}\langle O_p(\tau)O_p^\dagger(0)\rangle &= \langle P^\dagger O_p(\tau)O_p^\dagger(0)P\rangle \\ &= \langle (P^\dagger O_p P)(\tau)(P^\dagger O_p^\dagger(0)P)\rangle\end{aligned}$$

Under an elementary transposition, say P_{12}, we have

$$\begin{aligned}P_{12}R_x^pP_{12}|1\rangle &= P_{12}R_x^p|2\rangle = e^{\frac{4\pi ip}{q}}|1\rangle \\ P_{12}R_x^pP_{12}|2\rangle &= e^{\frac{2\pi ip}{q}}|2\rangle \\ P_{12}R_x^pP_{12}|n>2\rangle &= e^{\frac{2\pi inp}{q}}|n\rangle\end{aligned}$$

Thus

$$P_{12}R_x^pP_{12} = e^{\frac{4\pi ip}{q}}|1\rangle\langle 1| + e^{\frac{2\pi ip}{q}}|2\rangle\langle 2| + \sum_{n=2}^{q} e^{\frac{2\pi inp}{q}}|n\rangle\langle n|$$

which is some linear combination of all the $R_x^{p'}$. So we may write, in general,

$$P^\dagger O_p P = P^\dagger R_x^p P = \sum_{p'} G_{pp'}O_{p'}$$

Associated with every permutation P, we have a $q \times q$ matrix G. It is easy to see that the set of G matrices in fact form a q-dimensional representation of the permutation group. If we define $C_{pp'}(\tau) = \langle O_p(\tau) O^\dagger_{p'}(0) \rangle$, we have

$$C_{pp'}(\tau) = (GCG^\dagger)_{pp'}$$

This implies that $[G, C] = 0$ for every G. Now imagine decomposing the representation G into irreducible representations. This corresponds to going to a basis where each G is block diagonal with each block corresponding to a particular irrep. By Schur's lemma, C is diagonal in that basis with elements corresponding to the same block being equal. We proceed by proving the following claim.

Claim: G-rep. = (Identity irrep.) $\oplus$ ($(q-1)$-dim. irrep.)

Proof: $O_{p=0} = 1$ is invariant under the group action. Therefore G-rep. = (Identity rep.) $\oplus$ (Something). It is suffucient to show that there are exactly 2 irreps in the decomposition of G. To find the number of irreps, use character orthogonality:

$$\sum_{g \epsilon Gp} \chi^\mu(g) \chi^\nu(g^{-1}) = |Gp| \delta^{\mu\nu} \tag{C.1}$$

for μ, ν irreps, and $|Gp|$ is the order of the group. Thus, if G-rep $= \oplus n$-distinct irreps,

$$\sum_{g \epsilon Gp} \chi(g) \chi(g^{-1}) = n|Gp|$$

where $\chi_(g)$ are the characters in the representation G. The characters of G-rep can be calculated as follows:

$$\begin{aligned}
P^\dagger O_p P &= \sum_{p'} G_{pp'} O_{p'} \\
\frac{1}{q} tr(O^\dagger_r P^\dagger O_p P) &= \frac{1}{q} \sum_{p'} G_{pp'} tr(O^\dagger_r O_{p'}) = \frac{1}{q} \sum_{p'} G_{pp'} tr(R_x^{p'-r}) \\
&= \frac{1}{q} \sum_{p'} G_{pp'} \sum_n e^{i\frac{2\pi}{q} n(p'-r)} = G_{pr} \\
\chi(g) &= \sum_p G_{pp} = \frac{1}{q} \sum_{p=0}^{q-1} tr(O^\dagger_p P^\dagger O_p P)
\end{aligned}$$

If g corresponds to the permutation $n \rightarrow g_n$, then in $|n\rangle$ basis

$$P^\dagger O_p P |n\rangle = e^{2\pi i \frac{p}{q} g_n} |n\rangle$$

$$\begin{aligned}\chi(g) &= \frac{1}{q}\sum_{n}\sum_{p=0}^{q-1} e^{2\pi i \frac{p}{q}(g_n - n)} \\ &= \sum_{n} \delta_{n,g_n}\end{aligned}$$

A simple calculation then gives the result

$$\sum_{g} |\chi(g)|^2 = 2|Gp|$$

This proves the claim. The result that $C_{pp}(\tau)$ is independant of p for $p = 1, ..., q-1$ then follows from Schur's lemma.

Bibliography

[1] P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[2] A.F. Hebard and M.A. Paalanen, Phys. Rev. Lett. **65**, 927 (1990); M.P.A.Fisher, S.M. Girvin, and G. Grinstein, Phys. Rev. Lett. **64**, 587 (1990).

[3] C.L. Seaman et.al., Phys. Rev. Lett. **67**, 2882 (1991); B. Andraka and A.M. Tsvelik, Phys. Rev. Lett. **67**, 2886 (1991); H. von Lohneyson, J. Phys. **C 8**. 9689 (1996).

[4] S. Sachdev and J. Ye, Phys. Rev. Lett. **69**, 2411 (1992)

[5] B.M. McCoy and T.T. Wu, Phys. Rev. **176**, 631 (1968); **188**, 982 (1969)

[6] R. Shankar and G. Murthy, Phys. Rev. **B36**, 536 (1987).

[7] D.S. Fisher, Phys. Rev. Lett. **69**, 534 (1992); D.S. Fisher,Phys. Rev. **B51**, 6411 (1995).

[8] S.L. Sondhi, S.M. Girvin, J.P. Carini, and D. Shahar, Rev. Mod. Phys, **69**, 315, (1997).

[9] S. Sachdev, in *Proceedings of the 19th IUPAP International Conference on Statistical Physics*, Xiamen, China, 1995, edited by Hao Bailin (World Scientific, Singapore) p. 289 [preprint cond-mat/9508080]

[10] F.Y.Wu, Rev. Mod. Phys. **54**, 235 (1982)

[11] J. Jose, L. Kadanoff, S. Kirkpatrick, and D.R. Nelson, Phys. Rev. **B16**, 1217 (1977) ; S. Elitzur, R. Pearson, and J. Shigemitsu, Phys. Rev. **D19**, 3698 (1979)

[12] J.Solyom and P.Pfeuty, Phys. Rev. **B 24**, 218 (1981); L.Turban, J. Phys. A **18**, 2313 (1985)

[13] M.P.M. den-Nijs, J. Phys. **A12**, 1857 (1979).

[14] A.B. Harris, J. Phys. C **7**, 1671 (1974)

[15] J.T. Chayes, L. Chayes, D.S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).

[16] Y. Imry and M. Wortis, Phys. Rev. **B19**, 3581 (1979); K. Hui and A.N. Berker, Phys. Rev. Lett. **62**, 2507 (1989)

[17] M. Aizenman and J. Wehr, Phys. Rev. Lett., **62**, 2503 (1989)

[18] M.J. Thill and D.A. Huse, Physica **A15**, 321 (1995)

[19] N. Read, S. Sachdev, and J. Ye, Phys. Rev. **B52**, 384 (1995); J. Ye, S. Sachdev, and N. Read, Phys. Rev. Lett. **70**, 4011 (1993)

[20] M. Guo, R.N. Bhatt, and D.A. Huse, Phys. Rev. Lett. **72**, 4137 (1994); cond-mat/9605111.

[21] H. Rieger and A.P. Young, Phys. Rev. Lett. **72**, 4141 (1994); cond-mat/9512162.

[22] T. Senthil and S. Sachdev, Phys. Rev. Lett., **77**, 5292 (1996).

[23] R.B.Griffiths, Phys. Rev. Lett. **23**, 17 (1969)

[24] D. Dhar, in *Stochastic Processes: Formalism and Applications*, edited by G.S. Agarwal and S. Dattagupta (Springer, Berlin) 1983; M. Randeria, J.P. Sethna, and R.G. Palmer, Phys. Rev. Lett., **54**, 1321 (1985); D. Dhar, M. Randeria, and J.P. Sethna, Europhys. Lett.,**5**(6), 485 (1988); A.J. Bray, Phys. Rev. Lett., **60**, 720 (1988).

[25] See for e.g. N. Goldenfeld *Lectures on phase transitions and the renormalization group, Addison-Wesley (1992)* and references therein.

[26] For the quantum Ising and rotor models, this is done in detail in S. Sachdev, Phys. Rev. **B55**, 142 (1997)

[27] S.N. Dorogovstev, Phys. Lett., **76A**, 169 (1980)

[28] D. Boyanovsky and J.L. Cardy, Phys. Rev. **B26**, 154 (1982)

[29] Y.B. Kim and X.G. Wen, Phys. Rev. **B49**, 4052 (1994)

[30] S. Sachdev, unpublished

[31] A.P. Young and H. Rieger, cond-mat/9510027

[32] F.Y.Wu, Rev. Mod. Phys. **54**, 235 (1982)

[33] See for instance, R. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press (1982).

[34] S.K.Ma, C.Dasgupta, and C.-k.Hu, Phys.Rev. Lett. **43**, 1434 (1979); C.Dasgupta and S.K.Ma, Phys. rev. **B22**, 1305 (1980)

[35] This is well-known for the pure 1-d quantum systems; see Ref [32]. In the presence of randomness, under the duality transformation, the distributions of h and J get interchanged.

[36] C.A. Doty and D.S. Fisher, Phys. Rev. **B45**, 2167 (1992).

[37] S.Chen, A.M.Ferrenberg, and D.P.Landau, Phys. Rev. Lett., **69**, 1213 (1992)

[38] M.Kardar et. al., Phys. Rev. **E52**, R1269 (1995)

[39] J.Cardy, to appear in Phys. Rev. **E**

[40] A.W.W.Ludwig, Nucl. Phys. **B330**, 639 (1990)

[41] J. Cardy and J.L. Jacobsen, cond-mat/9705038

[42] D. Stauffer and A. Aharony, Introduction to Percolation Theory, Taylor and Francis Ltd. (1992)

[43] A.B. Harris, J. Phys. **C7**, 3082 (1974); R.B. Stinchcombe, J. Phys. **C14**, L263 (1981); R.R. dos Santos, J.Phys. **C15**, 3141 (1982).

[44] M.J. Stephen and G.S. Grest, Phys. Rev. Lett., **38**, 567 (1977).

[45] S. Sachdev, N. Read, and R. Oppermann, Phys. Rev. B **52**, 10286 (1995).

[46] A.B. Harris and T.C. Lubensky, Phys. Rev. **B24**, 2656 (1981)

[47] Y. Imry and S.-k. Ma, Phys. Rev. Lett. **35**, 1399 (1975)

[48] J.Z. Imbrie, Phys. Rev. Lett. **53**, 1747 (1984); Commun. Math. Phys. **98**, 145 (1985).

[49] See T.Nattermann, cond-mat/9705295 for a recent review of the theory of classical random field magnets.

[50] A. Aharony, Y. Imry and S.-k. Ma, Phys. Rev. Lett., **37**, 1364 (1976); G. Grinstein, Phys. Rev. Lett., **37**, 944 (1976).

[51] A.J. Bray and M.A. Moore, J. Phys. **C18**, L927 (1985)

[52] See for instance Newman et. al., Phys. rev. **B48**, 16533 (1993).

[53] J. Villain, J. de Phys., **46**, 1843 (1985); D.S. Fisher, Phys. Rev. Lett., **56**, 416 (1986)

[54] A. Aharony, Y. Gefen, and Y. Shapir, J. Phys. **C15**, 673 (1982)

[55] D. Boyanovsky and J.L. Cardy, Phys. Rev. **B27**, 5557 (1983).

[56] E. Pytte and Y. Imry, Phys. Rev. **B 35**, 1465 (1987)

[57] A.T. Ogielski and D.A. Huse, Phys. Rev. Lett. **56**, 1298 (1986)

[58] See D.P. Belanger, cond-mat/9706042 for a recent review of experiments on classical random field magnets.

[59] A.J.Bray and M.A.Moore, J. Phys. C **13**, L655 (1980)

[60] J.Miller and D.Huse, Phys. Rev. Lett. **70**, 3147 (1993)

[61] T. Senthil and S.N. Majumdar, Phys. Rev. Lett. **76**, 3001 (1996)

[62] See, for instance, T.C. Lubensky, in *Ill-Condensed Matter*, Les Houches 1978, R. Balian, R. Maynard and G. Toulouse (Eds.), North Holland/World Scientific 1983.

[63] T.R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. **74**, 1178 (1995).

[64] A.M. Sengupta and A. Georges, Phys. Rev. B **52**, 10295 (1995).

[65] S. Sachdev and N. Read, J. Phys. **C8**, 9723 (1996)

[66] I. Affleck, J. Phys. **C2**, 405 (1990).

[67] D.S. Fisher, Phys. Rev. B **50**, 3799 (1994).

[68] R.A. Hyman and K. Yang, Phys. Rev. Lett. **78**, 1783 (1997)

[69] T. Giamarchi and H. Schulz, Phys. Rev. B **37**, 325 (1988)

[70] W. Wu, B. Ellman, T.F. Rosenbaum, G. Aeppli, and D.H. Reich, Phys. Rev. Lett. **67**, 2076 (1991); W. Wu, D.Bitko, T.F. Rosenbaum, and G. Aeppli, Phys. Rev. Lett. **71**, 1919 (1993)